Univalent Monoidal Categories

Kobe Wullaert
Delft University of Technology, The Netherlands

Ralph Matthes
IRIT, Université de Toulouse, CNRS, Toulouse INP, UT3, Toulouse, France

Benedikt Ahrens
Delft University of Technology, The Netherlands
University of Birmingham, United Kingdom

Abstract

Univalent categories constitute a well-behaved and useful notion of category in univalent foundations. The notion of univalence has subsequently been generalized to bicategories and other structures in (higher) category theory. Here, we zoom in on monoidal categories and study them in a univalent setting. Specifically, we show that the bicategory of univalent monoidal categories is univalent. Furthermore, we construct a Rezk completion for monoidal categories: we show how any monoidal category is weakly equivalent to a univalent monoidal category, universally. We have fully formalized these results in UniMath, a library of univalent mathematics in the Coq proof assistant.

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Contents

1 Introduction

When working in univalent foundations (see [15]), definitions have to be designed carefully in order to correspond, via the intended semantics, to the *expected* notions in set-theoretic foundations. The notion of univalent category [3] has been shown to be a good notion, in the sense that it corresponds to the usual notion of category under Voevodsky's model in simplicial sets [9].[1] Examples of univalent categories are plentiful, but not all categories arising in practice—for instance when studying categorical semantics of type theory—are univalent. In [3], the authors give a construction of a "free" univalent category from any category $\mathcal{C}$, which they call the Rezk completion of $\mathcal{C}$.

Since then, the univalence condition and completion operation have been studied further.

Firstly, in [16], Van der Weide constructs a class of higher inductive types using the groupoid quotient. It is shown that the groupoid quotient gives rise to a biadjunction between the bicategory of groupoids and the bicategory of 1-types (which is isomorphic to the bicategory of univalent groupoids); the left adjoint thus yields a univalent completion operation for groupoids. Van der Weide furthermore lifts this completion to "structured groupoids", that is, to groupoids equipped with an algebra structure for some endo-pseudofunctor on (univalent) groupoids.

Secondly, the univalence condition on categories was extended to bicategories in [2] and to other (higher-)categorical structures in [5]. In more detail, [5] develops a notion of theory for mathematical structures, and a notion of univalence for models of such theories.

Thirdly, univalent displayed graphs are used in [6] to define and study higher groups.

In the present paper, we continue the study of univalent (higher-)categorical structures, focusing on *monoidal* categories. Monoidal categories are very useful in a variety of contexts, such as quantum mechanics [8] and computing [7], modeling concurrency [11], probability theory [13] and probabilistic programming [12], and neural networks [10]. We present two results on monoidal categories:

1. We show that the bicategory of univalent monoidal categories is univalent. Here, a univalent monoidal category is a univalent category with a monoidal structure.
2. We construct, for any monoidal category, a monoidal Rezk completion. It is, in particular, a univalent monoidal category; the challenge lies in establishing the universal property of a Rezk completion, here modified for monoidal categories.

Both results have been formalized in the `UniMath` library of univalent mathematics, based on the Coq proof assistant.

The first of these results may be considered to be a basic sanity check; failing to prove this would question the validity of our definitions. However, its proof is technically difficult, and, in our experience, only feasible through the disciplined application of "displayed" technology as developed in [4] and [2].

The second result consists, more specifically, of a lifting of the Rezk completion for categories as constructed in [3] to the monoidal structure. As such, it also relies on displayed technology: the equivalence expressing the universal property of our monoidal Rezk completion is given as a displayed equivalence on top of the equivalence constructed in [3].

Our work is strongly related to some of the work mentioned above.

[1] To emphasize that univalent categories are the right notion of category in univalent foundations, they are just called "categories" in [3].

Firstly, an instance of Van der Weide's work covers monoidal groupoids; see [16, Section 6.7.4]. Compared to that work, our work discusses monoidal *categories* rather than groupoids, but does not cover general structures. In particular, we also provide a completion operation for *lax* and *oplax* monoidal categories. Work on the "pushout" of our and Van der Weide's work, a Rezk completion for structured categories, is ongoing (see also Section 5).

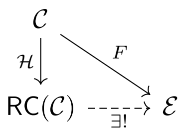
Secondly, [5, Example 8.7] studies monoidal categories. It is shown there that the general univalence condition on a model of the theory of monoidal categories defined in that work simplifies, in the case of monoidal categories, to the underlying category being univalent. Thus, the univalent monoidal categories of [5, Example 8.7] are the same as the ones studied in the present work.

In the remainder of the introduction, we review the Rezk completion and displayed (bi)categories, respectively. We also give some details about the formalization.

▶ **Notation 1.** *In order to stay consistent with the notation used in* `UniMath`*, we write the composition in diagrammatic order, i. e., the composition of* $f : x \to y$ *and* $g : y \to z$ *is denoted as* $f \cdot g : x \to z$*.*

1.1 Review of the Rezk Completion for Categories

The Rezk completion for categories was constructed in [3]. In essence, given a category $\mathcal{C}$, its Rezk completion is given by a univalent category $\mathsf{RC}(\mathcal{C})$ and a weak equivalence $\mathcal{H} : \mathcal{C} \to \mathsf{RC}(\mathcal{C})$. It has the following property: any functor $F : \mathcal{C} \to \mathcal{E}$, with $\mathcal{E}$ a univalent category, factors *uniquely* via $\mathcal{H}$, as depicted in the following diagram.



(1)

▶ Remark 2. The universal property satisfied by the Rezk completion is a bicategorical one, see Definition 4. From a purely category-theoretic viewpoint, the factorization in Equation (1) is unique up to natural isomorphism. However, since $\mathcal{E}$ is univalent, the functor category $[\mathsf{RC}(\mathcal{C}), \mathcal{E}]$ is also univalent. Therefore, the lifting of a functor is unique.

In [3], it is said that the construction gives a universal way to replace a category by a univalent category. This construction is indeed universal in a bicategorical sense, according to the following lemma:

▶ **Lemma 3** ([3, Thm. 8.4], `precomp_adjoint_equivalence`)**.** *Let* $\mathcal{H} : \mathcal{C} \to \mathcal{D}$ *be a weak equivalence between categories. For any univalent category* $\mathcal{E}$*, the functor* $\mathcal{H} \cdot (-) : [\mathcal{D}, \mathcal{E}] \to [\mathcal{C}, \mathcal{E}]$ *is an adjoint equivalence of categories.*

In order to see that the Rezk completion is indeed a universal object, recall that a biadjunction can be expressed as a certain adjoint equivalence of hom-categories:

▶ **Definition 4** (`left_universal_arrow`)**.** *A pseudo-functor* $R : \mathcal{B}_2 \to \mathcal{B}_1$ *has a left biadjoint if for any object* $x : (\mathcal{B}_1)_0$ *there is a* ***left universal arrow from*** x ***to*** R*:*

1. *an object* $L\,x : (\mathcal{B}_2)_0$*,*
2. *a morphism* $\eta_x : \mathcal{B}_1(x, R(L\,x))$*;*
3. *for any* $y : (\mathcal{B}_2)_0$*, the functor*

$$\eta_x \cdot (R\,-) : \mathcal{B}_2(L\,x, y) \to \mathcal{B}_1(x, R\,y) \ ,$$

whose action on objects is $f \mapsto \eta_x \cdot R\,f$ *and whose action on morphisms is* $\alpha \mapsto \eta_x \triangleleft R\,\alpha$*, is an adjoint equivalence of categories.*

Therefore, Lemma 3 means precisely that the Rezk completion of a category $\mathcal{C}$ gives a left universal arrow from $\mathcal{C}$ to the forgetful functor from the bicategory $\mathbf{Cat}_{Univ}$ of univalent categories, functors, and natural transformations to the bicategory **Cat** of categories, functors, and natural transformations.

1.2 Review of Displayed (Bi)Categories

In this section, we recall the basic concepts of displayed bicategories and their univalence. More information can be found in [1].

Let us first briefly recall the idea of displayed categories.

Many concrete examples of categories are given by structured sets and structure-preserving functions. An example of this is the category **Mon** of monoids and monoid homomorphisms. In particular, an identity morphism is an identity function (i. e., the identity morphism in **Set**) and the composition of monoid homomorphisms is given by the composition of the underlying functions (i. e., the composition in **Set**). Therefore, working in a category of structured sets often means lifting structure of the category **Set** to the additional structure. An example of this phenomenon is the product of monoids: the underlying set of a product of monoids can be constructed as the product of the underlying sets (Example 5).

The notion of **displayed category** formalizes the process of creating a new category out of an old category by adding structure and/or properties on the objects and/or morphisms in the following way: a displayed category ([4, Def. 3.1]) specifies precisely the extra structure and the extra laws needed to build the new category out of the old one. This new category is then called the *total category* of the displayed category ([4, Def. 3.2]).

▶ **Example 5.** The category **Mon** of monoids can be constructed as a total category over **Set** as follows:

1. For $X : \mathbf{Set}$, the type of displayed objects over X is the type of monoid structures on X:

$$\sum_{m:X\times X\to X}\sum_{e:X}\mathsf{isAssociative}(m)\times\prod_{x:X}(e\cdot x = x\times x\cdot e = x),$$

   where $\mathsf{isAssociative}(m)$ is the proposition witnessing that m is associative.

2. Assume given $X, Y : \mathbf{Set}$, $f : \mathbf{Set}(X,Y)$ and (m_X, e_X, p_X) (resp. (m_Y, e_Y, p_Y)) displayed object over X (resp. Y), i. e., the structure of a monoid. The type of displayed morphisms over f is the proposition witnessing that f is a monoid homomorphism from (m_X, e_X, p_X) to (m_Y, e_Y, p_Y):

$$(f\,e_X = e_Y)\times\prod_{x_1,x_2:X} f\,(m_X(x_1,x_2)) = m_Y(f\,x_1, f\,x_2).$$

Analogously, there is also the notion of a **displayed bicategory**:

▶ **Definition 6** ([1, Def. 6.1], `disp_bicat`)**.** *Let $\mathcal{B}$ be a bicategory. A* ***displayed bicategory*** *$\mathcal{D}$ over $\mathcal{B}$ consists of:*

1. *for any $x : \mathcal{B}$, a type $\mathcal{D}_x$ of* displayed objects *over x,*
2. *for any $f : \mathcal{B}(x,y)$ and $\bar{x} : \mathcal{D}_x$ and $\bar{y} : \mathcal{D}_y$, a type $\mathcal{D}_f(\bar{x},\bar{y})$ of* displayed morphisms *over f,*
3. *for any $\alpha : \mathcal{B}(x,y)(f,g)$ and $\bar{f} : \mathcal{D}_f(\bar{x},\bar{y})$ and $\bar{g} : \mathcal{D}_g(\bar{x},\bar{y})$, a set $\bar{f} \overset{\alpha}{\Rightarrow} \bar{g}$ of* displayed 2-cells *over α;*

together with a composition of displayed morphisms and displayed 2*-cells (over the composition in $\mathcal{B}$) and a displayed identity morphism and* 2*-cell (over the identity morphism resp.* 2*-cell in $\mathcal{B}$). The axioms of a bicategory have corresponding displayed axioms (over those axioms in $\mathcal{B}$).*

▶ **Definition 7** ([1, Def. 6.2], `total_bicat`). *Let $\mathcal{D}$ be a displayed bicategory over $\mathcal{B}$. The **total bicategory** of $\mathcal{D}$, denoted as $\int \mathcal{D}$, has as i-cells (with $i = 0, 1, 2$), pairs $(x, \bar{x})$ where x is an i-cell of $\mathcal{B}$ and $\bar{x}$ is a displayed i-cell of $\mathcal{D}$ over x.*

▶ **Example 8.** The bicategory whose objects are categories equipped with a terminal object, whose morphisms are functors preserving the terminal objects (strongly) and whose 2-cells are natural transformations, can be constructed as the total bicategory over **Cat** as follows:

1. For $\mathcal{C}$: **Cat**, the type of displayed objects over **Cat** is the type witnessing that $\mathcal{C}$ has a terminal object:
$$\sum_{X:\mathcal{C}} \mathsf{isTerminal}(X).$$
2. Assume given $\mathcal{C}, \mathcal{D} : \mathbf{Cat}$, $F : \mathbf{Cat}(\mathcal{C}, \mathcal{D})$ and $(T_{\mathcal{C}}, p_{\mathcal{C}})$ (resp. $(T_{\mathcal{D}}, p_{\mathcal{D}})$) displayed objects over $\mathcal{C}$ (resp. $\mathcal{D}$). The type of displayed morphisms over F is the proposition witnessing that F preserves the terminal object:
$$\mathsf{isIsomorphism}(!),$$
where ! is the unique morphism $F\,T_{\mathcal{C}} \to T_{\mathcal{D}}$ given by the universal property of the terminal object $T_{\mathcal{D}}$.
3. Let $F, G : \mathbf{Cat}(\mathcal{C}, \mathcal{D})$ be functors between categories $\mathcal{C}$ and $\mathcal{D}$ and assume:
   a. $(T_{\mathcal{C}}, p_{\mathcal{C}})$ (resp. $(T_{\mathcal{D}}, p_{\mathcal{D}})$) a witness that $\mathcal{C}$ (resp. $\mathcal{D}$) has a terminal object, i. e., it is a displayed object over $\mathcal{C}$ (resp. $\mathcal{D}$),
   b. μ^F (resp. μ^G) a proof witnessing that F (resp. G) preserves the terminal object strongly, i. e., μ^F (resp. μ^G) is a displayed morphism over F (resp. G).

   For any natural transformation $\alpha : F \Rightarrow G$, the type of displayed 2-cells is the unit type.

Given displayed bicategories $\mathcal{D}_1$ and $\mathcal{D}_2$ over a bicategory $\mathcal{B}$, we construct the product $\mathcal{D}_1 \times \mathcal{D}_2$ over $\mathcal{B}$. The displayed objects, morphisms, and 2-cells are pairs of objects, morphisms, and 2-cells, respectively (`disp_dirprod_bicat`).

A displayed bicategory is *locally* (resp. *globally*) *univalent* if the canonical function from the equality type of displayed morphisms (resp. displayed objects) to the type of displayed isomorphisms (resp. displayed adjoint equivalences) is an equivalence of types. A displayed bicategory is *univalent* if it is both locally and globally univalent (`disp_univalent_2`, `disp_univalent_2_0`, `disp_univalent_2_1`).

▶ **Lemma 9** ([1, Thm. 7.4], `total_is_univalent_2`). *Let $\mathcal{D}$ be a displayed bicategory over $\mathcal{B}$ and $q \in \{$locally, globally$\}$. Then $\int D$ is q-univalent if $\mathcal{B}$ is q-univalent and $\mathcal{D}$ is q-univalent.*

▶ Remark 10. As witnessed by Lemma 9, certain properties of the total bicategory can be expressed in terms of the *base* bicategory and the displayed bicategory. This allows one to divide a problem, in this case showing univalence, into multiple steps.

Therefore, while we are interested in studying the total bicategory, we usually only describe the displayed bicategory.

1.3 Formalization in UniMath

The results presented here are formulated inside intensional dependent type theory. We carefully distinguish between data and properties, i. e., data is always explicitly given which avoids the use of the axiom of choice and the law of excluded middle. The results presented here are formalized and checked in the library `UniMath` of univalent mathematics, based on the proof assistant `Coq` [14].

The formalization referred to in this paper is presented in the `UniMath` commit `6d2d288`. An HTML documentation of this commit is hosted online. Most of our definitions, lemmas, and theorems are accompanied by a link which leads to the corresponding definition, lemma, and theorem in the documentation.

The formalization is built upon the existing library of (bi)category theory and the theory of displayed (bi)categories. The (1-)categorical formulation of displayed categories has been developed in [4] and the bicategorical formulation has been developed in [2].

The accompanying code, specific to this work, consists of approximately 7000 lines of code. However, the formalisation also made it necessary to contribute to the `UniMath` library on monoidal categories more generally.

2 The Bicategory of Monoidal Categories

In this section we construct the bicategory **MonCat** (resp. $\mathbf{MonCat}^{stg}$) of monoidal categories, lax (resp. strong) monoidal functors and monoidal natural transformations. We construct this bicategory as the total bicategory of a displayed bicategory over the bicategory **Cat** of categories, functors, and natural transformations.

This displayed bicategory in itself is constructed by stacking different displayed bicategories. This can indeed be done because, e. g., the tensor product and unit object can be defined independently from, e. g., the unitors.

▶ Remark 11. Although the construction of **MonCat** (resp. $\mathbf{MonCat}^{stg}$) is standard (when working in univalent foundations), we explain the construction in quite some detail because both Section 3 and Section 4 heavily depend on the construction of monoidal categories (resp. lax/strong monoidal functors and natural transformations) in this displayed way. In particular, this allows us to fix notation and allows for the big picture of the constructions to become more visible.

The first displayed bicategory we construct adds the structure of a tensor and a unit. Since the unit and tensor are (without the unitors) independent of each other, we can define this as the product of displayed bicategories, the first representing the tensor and the second representing the unit.

▶ **Definition 12** (`bidisp_tensor_disp_bicat`). *The displayed bicategory* $\mathbf{Cat}_T$ *over* **Cat** *is defined as follows:*

1. *The displayed objects over a category* $\mathcal{C} : \mathbf{Cat}$ *are the functors of type* $\mathcal{C} \times \mathcal{C} \to \mathcal{C}$*, called* tensors *over* $\mathcal{C}$ *and are denoted by* $\otimes_{\mathcal{C}}$*.*
2. *The displayed morphisms over a functor* $F : \mathcal{C} \to \mathcal{D}$ *from* $\otimes_{\mathcal{C}}$ *to* $\otimes_{D}$ *are the natural transformations of type* $(F \times F) \cdot \otimes_{\mathcal{D}} \Rightarrow \otimes_{\mathcal{C}} \cdot F$*, called* witnesses of tensor-preservation of F*.*
3. *The displayed* 2*-cells over a natural transformation* $\alpha : F \Rightarrow G$ *from* μ^F *to* μ^G *are the proofs of the proposition*

$$\prod_{x,y:\mathcal{C}} (\alpha_x \otimes_D \alpha_y) \cdot \mu^G_{x,y} = \mu^F_{x,y} \cdot \alpha_{x \otimes_C y} \quad .$$

▶ **Definition 13** (`bidisp_unit_disp_bicat`). *The displayed bicategory* $\mathbf{Cat}_U$ *over* **Cat** *is defined such that:*

1. *The displayed objects over a category $\mathcal{C} : \mathbf{Cat}$ are the objects of $\mathcal{C}$, called* units *over $\mathcal{C}$ and are denoted by $I_{\mathcal{C}}$.*
2. *The displayed morphisms over a functor $F : \mathcal{C} \to \mathcal{D}$ from $I_{\mathcal{C}}$ to $I_{\mathcal{D}}$ are the morphisms of type $\mathcal{D}(I_{\mathcal{D}}, FI_{\mathcal{C}})$, called* witnesses of unit-preservation of F.
3. *The displayed 2-cells over a natural transformation $\alpha : F \Rightarrow G$ from ϵ^F to ϵ^G are the proofs of the proposition*

$$\epsilon^F \cdot \alpha_{I_{\mathcal{C}}} = \epsilon^G \ .$$

We denote by $\mathbf{Cat}_{TU}$ the displayed bicategory which is the product of $\mathbf{Cat}_T$ and $\mathbf{Cat}_U$ (`bidisp_tensor_unit`).

To fix some notation: The total bicategory $\int \mathbf{Cat}_{TU}$ has as objects triples $(\mathcal{C}, \otimes_{\mathcal{C}}, I_{\mathcal{C}})$ where $\mathcal{C}$ is a category, $\otimes_{\mathcal{C}}$ a tensor on $\mathcal{C}$ and $I_{\mathcal{C}}$ a unit on $\mathcal{C}$. A morphism from $(\mathcal{C}, \otimes_{\mathcal{C}}, I_{\mathcal{C}})$ to $(\mathcal{D}, \otimes_{\mathcal{D}}, I_{\mathcal{D}})$ is a triple (F, μ^F, ϵ^F) where F is a functor of type $\mathcal{C} \to \mathcal{D}$, μ^F a witness of tensor-preservation of F and ϵ^F a witness of unit-preservation of F.

We now add the unitors and the associator. Since they are independent of each other (before adding the triangle and pentagon equalities), we can again define them as a product of displayed bicategories. These displayed bicategories have trivial displayed 2-cells since monoidal natural transformations only use the data of the tensor and the unit. Thus we define these displayed bicategories as displayed categories. The formal construction of turning a displayed category into a displayed bicategory with trivial 2-cells is formalized as `disp_cell_unit_bicat`.

▶ **Definition 14** (`bidisp_lu_disp_bicat`)**.** *The displayed bicategory $\mathbf{Cat}_{LU}$ over $\int \mathbf{Cat}_{TU}$ is defined as the displayed category (with trivial 2-cells) such that:*

1. *The displayed objects over a triple $(\mathcal{C}, \otimes_C, I_C)$ are the natural isomorphisms of type $\mathbf{Cat}(I_{\mathcal{C}} \otimes_{\mathcal{C}} -, \mathsf{Id}_{\mathcal{C}})$, called* left unitors *over $(\mathcal{C}, \otimes_C, I_C)$ and are denoted by $\lambda^{\mathcal{C}}$.*
2. *The displayed morphisms over a triple (F, μ^F, ϵ^F) from $\lambda^{\mathcal{C}}$ to $\lambda^{\mathcal{D}}$ are proofs of the proposition:*

$$\prod_{x:\mathcal{C}} (\epsilon^F \otimes_{\mathcal{D}} \mathsf{Id}_{Fx}) \cdot \mu^F_{I_{\mathcal{C}},x} \cdot F\lambda^{\mathcal{C}}_x = \lambda^{\mathcal{D}}_{Fx} \ .$$

▶ **Definition 15** (`bidisp_ru_disp_bicat`)**.** *The displayed bicategory $\mathbf{Cat}_{RU}$ over $\int \mathbf{Cat}_{TU}$ is defined as the displayed category (with trivial 2-cells) such that:*

1. *The displayed objects over a triple $(\mathcal{C}, \otimes_C, I_C)$ are the natural isomorphisms of type $\mathbf{Cat}(- \otimes_{\mathcal{C}} I_{\mathcal{C}}, \mathsf{Id}_{\mathcal{C}})$, called* right unitors *over $(\mathcal{C}, \otimes_C, I_C)$ and are denoted as $\rho^{\mathcal{C}}$.*
2. *The displayed morphisms over a triple (F, μ^F, ϵ^F) from $\rho^{\mathcal{C}}$ to $\rho^{\mathcal{D}}$ are proofs of the proposition:*

$$\prod_{x:\mathcal{C}} (\mathsf{Id}_{Fx} \otimes_{\mathcal{D}} \epsilon^F) \cdot \mu^F_{x,I_{\mathcal{C}}} \cdot F\rho^{\mathcal{C}}_x = \rho^{\mathcal{D}}_{Fx} \ .$$

▶ **Definition 16** (`bidisp_associator_disp_bicat`)**.** *The displayed bicategory $\mathbf{Cat}_A$ over $\int \mathbf{Cat}_{TU}$ is defined as the displayed category (with trivial 2-cells) such that:*

1. *The displayed objects over a triple $(\mathcal{C}, \otimes_C, I_C)$ are the natural isomorphisms of type $\mathbf{Cat}((- \otimes_{\mathcal{C}} -) \otimes_{\mathcal{C}} -, - \otimes_{\mathcal{C}} (- \otimes_{\mathcal{C}} -))$, called* associators *over $(\mathcal{C}, \otimes_C, I_C)$ and are denoted as $\alpha^{\mathcal{C}}$.*

2. *The displayed morphisms over a triple* (F, μ^F, ϵ^F) *from* $\alpha^{\mathcal{C}}$ *to* $\alpha^{\mathcal{D}}$ *are proofs of the proposition:*

$$\prod_{x,y,z:\mathcal{C}} (\mu^F_{x,y} \otimes_{\mathcal{D}} \mathsf{Id}_{Fz}) \cdot \mu^F_{x \otimes_{\mathcal{C}} y,z} \cdot F\alpha^{\mathcal{C}}_{x,y,z} = \alpha^{\mathcal{D}}_{Fx,Fy,Fz} \cdot (\mathsf{Id}_{Fx} \otimes_{\mathcal{D}} \mu^F_{y,z}) \cdot \mu^F_{x,y \otimes_{\mathcal{C}} z} \ .$$

We denote by $\mathbf{Cat}_{UA}$ the displayed bicategory over $\int \mathbf{Cat}_{TU}$ which is the product of $\mathbf{Cat}_{LU}, \mathbf{Cat}_{RU}$ and $\mathbf{Cat}_A$ (`bidisp_assunitors_disp_bicat`).

▶ **Definition 17** (`disp_bicat_univmon`). *The displayed bicategory* $\mathbf{Cat}_P$ *is the full displayed sub-bicategory of* $\mathbf{Cat}_{UA}$ *specified by the product of the following predicates:*

1. *Triangle equality:*

$$\prod_{x,y:\mathcal{C}} \alpha_{x,I,y} \cdot \mathsf{Id}_x \otimes \lambda_y = \rho_x \otimes \mathsf{Id}_y \ .$$

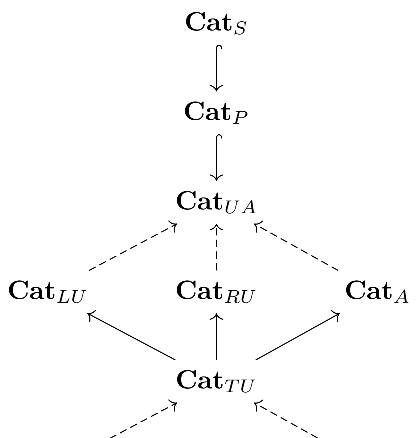
2. *Pentagon equality:*

$$\prod_{w,x,y,z:\mathcal{C}} (\alpha_{w,x,y} \otimes \mathsf{Id}_z) \cdot \alpha_{w,x\otimes y,z} \cdot \mathsf{Id}_w \otimes \alpha_{x,y,z} = \alpha_{w\otimes x,y,z} \cdot \alpha_{w,x,y\otimes z} \ .$$

▶ **Definition 18** (`disp_bicat_univstrongfunctor`). *The displayed bicategory* $\mathbf{Cat}_S$ *is the (non-full) displayed sub-bicategory of* $\mathbf{Cat}_P$ *where the displayed morphisms are proofs of the proposition*

$$\mathsf{isIso}(\epsilon) \times \prod_{x,y:\mathcal{C}} \mathsf{isIso}(\mu_{x,y}) \ .$$

The bicategory of monoidal categories, lax (resp. strong) monoidal functors, and monoidal natural transformations is denoted by $\mathbf{MonCat} := \int \mathbf{Cat}_P$ (resp. $\mathbf{MonCat}^{stg} := \int \mathbf{Cat}_S$). Their constructions are summarized in the following figure:



▶ Remark 19. An object in $\mathbf{MonCat}$ is of the form $(((\mathcal{C}, \otimes, I), \lambda, \rho, \alpha), tri, pent)$. In this form, however, it is not immediate (for, e. g., a proof assistant) that such an object is a category with extra structure. Therefore we consider in the formalization not $\mathbf{MonCat}$ as defined above, but we have applied the sigma construction (`sigma_bicat`) in order for an object to be of the form $(\mathcal{C}, (((\otimes, I), \lambda, \rho, \alpha), tri, pent))$. Switching between these bicategories does not change the overall message of this paper, although there are some extra steps that we have to take in order to conclude that the constructed displayed bicategory in this way is univalent.

Another difference with the formalization is that in the formalization of $\mathbf{Cat}_{LU}$ (resp. $\mathbf{Cat}_{RU}$, $\mathbf{Cat}_{A}$), we do not yet require a left unitor (resp. right unitor, associator) to be an isomorphism. Since being an isomorphism is a proposition, we could and did add these three (indexed) conditions only in the formalization of $\mathbf{Cat}_{P}$.

In Section 4, we construct a Rezk completion for monoidal categories. We are interested in studying the hom-categories of **MonCat** and thus, in particular, the displayed hom-categories. We now introduce some notations. Let $\mathcal{B}$ be a bicategory and $x, y : \mathcal{B}$ objects. The hom-category from x to y is denoted by $\mathcal{B}(x, y)$. Any morphism $f : \mathcal{B}(x, y)$ induces a functor between hom-categories, more precisely:

▶ **Definition 20.** *Let $\mathcal{B}$ be a bicategory, $f : \mathcal{B}(x, y)$ a morphism and $z : \mathcal{B}$ an object. The* ***functor given by precomposition with f and target object z*** *is the functor*

$$f \cdot (-) : \mathcal{B}(y, z) \to \mathcal{B}(x, z) \ ,$$

where the action on the objects is given by precomposition, i. e., $g \mapsto f \cdot g$, and the action on the morphisms is given by left whiskering, i. e., $\alpha \mapsto f \triangleleft \alpha$.

We also refer to the functor given by precomposition with f as the **precomposition functor with f**.

Let $\mathcal{D}$ be a displayed bicategory over $\mathcal{B}$ and $\bar{x} \in \mathcal{D}_x$ and $\bar{y} \in \mathcal{D}_y$ be displayed objects. The (total) hom-category $\int \mathcal{D}((x, \bar{x}), (y, \bar{y}))$ can be constructed as a total category of a displayed category over $\mathcal{B}(x, y)$. We denote this displayed category by $\mathcal{D}(\bar{x}, \bar{y})$ (so we use the same notation for the hom-categories and displayed hom-categories).

In particular, the precomposition functor w. r. t. the total bicategory $\int \mathcal{D}$ of a morphism $(f, \bar{f})$ can be defined as a displayed functor over the precomposition functor $f \cdot (-)$ (w. r. t. $\mathcal{B}$) where we precompose/left whisker (in the displayed sense) with $\bar{f}$:

▶ **Definition 21.** *Let $\mathcal{D}$ be a displayed bicategory over a bicategory $\mathcal{B}$, $\bar{x} : \mathcal{D}_x, \bar{y} : \mathcal{D}_y$ displayed objects, $\bar{f} : \mathcal{D}_f(\bar{x}, \bar{y})$ a displayed morphism and $\bar{z} : \mathcal{D}_z$ a displayed object. The* ***displayed functor given by precomposition with $\bar{f}$ and target displayed object $\bar{z}$*** *is the displayed functor*

$$\bar{f} \cdot (-) : \mathcal{D}(\bar{y}, \bar{z}) \to \mathcal{D}(\bar{x}, \bar{z})$$

over the functor given by precomposition with f.

We also refer to the displayed functor given by precomposition with $\bar{f}$ as the **displayed precomposition functor with $\bar{f}$**.

3 The Univalent Bicategory of Monoidal Categories

In this section we present our proof of the fact that the bicategory of univalent monoidal categories is univalent. In this proof, we rely heavily on the *displayed* machinery built in [2], for modular construction of bicategories, and proofs of their univalence.

In the formalization of this univalence proof, we have not used the formalization of a monoidal category as presented above. Instead, we have changed the definition of a tensor from being a functor to a more explicit, unfolded definition. It is not necessarily obvious that the resulting bicategory is indeed that of monoidal categories, lax (resp. strong) monoidal functors, and monoidal natural transformations. Therefore, we construct an equivalence of types of monoidal categories as presented above on the one hand and using this explicit definition on the other hand (`cmonoidal_to_noncurriedmonoidal`, `cmonoidal_adjequiv_noncurried_hom`).

Recall from Lemma 9 that the total bicategory of a displayed bicategory is univalent if the base bicategory is univalent and the displayed bicategory is univalent. Since $\mathbf{Cat}_{Univ}$ is univalent, it therefore reduces to showing that $\Sigma_{\Sigma_{\mathbf{Cat}_{TU}} \mathbf{Cat}_{UA}} \mathbf{Cat}_P$ from the previous section is univalent. (This is to be read modulo the repackaging described in Remark 19.)

The sigma construction of univalent displayed bicategories is univalent provided that both displayed bicategories are locally groupoidal (i. e., all displayed 2-cells are invertible) and locally propositional (i. e., each type of displayed 2-cells is a proposition). The previously defined displayed bicategories are locally propositional since they either express an (indexed) equality of morphisms or the type of 2-cells is the unit type. Thus in this section, we show that the displayed bicategories from Section 2 are univalent and locally groupoidal.

▶ Remark 22. In this section we implicitly restrict the displayed bicategories to the bicategory $\mathbf{Cat}_{Univ}$ of univalent categories, e. g., $\mathbf{Cat}_{TU}$ is considered as the pullback of the displayed bicategory $\mathbf{Cat}_{TU}$ along the inclusion of $\mathbf{Cat}_{Univ}$ into $\mathbf{Cat}$.

▶ **Lemma 23** (`tensor_disp_is_univalent_2`)**.** $\mathbf{Cat}_T$ *is univalent.*

Proof. $\mathbf{Cat}_T$ is locally univalent by a straightforward calculation, we therefore only discuss that it is globally univalent.

Let $\otimes_1, \otimes_2$ be two tensors on $\mathcal{C}$. We have to show that $\mathsf{idtoiso}^{2,0}_{\otimes_1,\otimes_2}$ is an equivalence of types. In order to show this, we factorize this function as follows:

$$\begin{array}{ccc} \otimes_1 = \otimes_2 & \xrightarrow{\mathsf{idtoiso}^{2,0}_{\otimes_1,\otimes_2}} & \mathsf{DispAdjEquiv}(\otimes_1, \otimes_2) \\ {\scriptstyle \mathsf{idtoeq}}\downarrow & & \uparrow \\ \mathsf{tensorEq}(\otimes_1, \otimes_2) & \xrightarrow[\mathsf{eqtoiso}]{} & \mathsf{tensorIso}(\otimes_1, \otimes_2) \end{array} ,$$

where $\mathsf{tensorEq}(\otimes_1, \otimes_2)$ is the type

$$\sum\nolimits_{\alpha : \prod_{x,y:\mathcal{C}}, x \otimes_1 y = x \otimes_2 y} \prod_{f:\mathcal{C}(x_1,y_1), g:\mathcal{C}(x_2,y_2)} f \otimes_1 g = f \otimes_2 g \ ,$$

where the equality $f \otimes_1 g = f \otimes_2 g$ is dependent over α_{x_1,y_1} and α_{x_2,y_2}.

The type $\mathsf{tensorIso}(\otimes_1, \otimes_2)$ is the same as $\mathsf{tensorEq}(\otimes_1, \otimes_2)$ where we replaced the first equality by an isomorphism (and the dependent equality of morphisms is replaced by pre- and post-composing with the isomorphism).

The function $\mathsf{idtoeq} : \otimes_1 = \otimes_2 \to \mathsf{tensorEq}(\otimes_1, \otimes_2)$ *forgets* the proofs of the properties of the tensor. Because our hom-types are sets, this is an equivalence. The function $\mathsf{eqtoiso} : \mathsf{tensorEq}(\otimes_1, \otimes_2) \to \mathsf{tensorIso}(\otimes_1, \otimes_2)$ exists because each identity induces an isomorphism. Since $\mathcal{C}$ is a univalent category, it is indeed an equivalence. Since a displayed adjoint equivalence in $\mathbf{Cat}_T$ translates into the notion of $\mathsf{tensorIso}(\otimes_1, \otimes_2)$, we construct in a straightforward manner a function from $\mathsf{tensorIso}(\otimes_1, \otimes_2)$ to $\mathsf{DispAdjEquiv}(\otimes_1, \otimes_2)$, which is for the same reason an equivalence. ◀

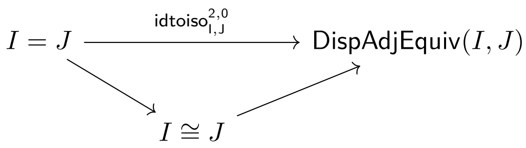
▶ **Lemma 24** (`tensor_disp_locally_groupoidal`)**.** $\mathbf{Cat}_T$ *is locally groupoidal.*

Proof. $\mathbf{Cat}_T$ being locally groupoidal means that if a natural isomorphism α preserves the tensor, then so does its inverse. This is immediate since the tensor product of isomorphisms is again an isomorphism (by functoriality of the tensor). ◀

▶ **Lemma 25** (`unit_disp_is_univalent_2`). $\mathbf{Cat}_U$ *is univalent.*

Proof. $\mathbf{Cat}_U$ is locally univalent by a straightforward calculation. Therefore, we only discuss why it is globally univalent.

Let $I, J : \mathcal{C}$ be objects representing a unit object. As with the tensor layer, we factorize $\mathsf{idtoiso}^{2,0}_{\mathsf{I},\mathsf{J}}$ and show that each function in the factorization is an equivalence. The factorization is given by:



The definition of a displayed adjoint equivalence in this displayed bicategory translates precisely to an isomorphism in the underlying category $\mathcal{C}$, which gives us the arrow to the right and a proof that it is an equivalence. The left arrow is given by $\mathsf{idtoiso}_{I,J}$ and is an equivalence precisely because $\mathcal{C}$ is a univalent category. ◀

▶ **Lemma 26** (`unit_disp_locally_groupoidal`). $\mathbf{Cat}_U$ *is locally groupoidal.*

▶ **Lemma 27** (`assunitors_disp_is_univalent_2`). $\mathbf{Cat}_{UA}$ *is univalent.*

Proof. Since the product of univalent displayed bicategories is univalent, it remains to show that $\mathbf{Cat}_{LU}, \mathbf{Cat}_{RU}$ and $\mathbf{Cat}_A$ are univalent.

These displayed bicategories are locally univalent because the type of (displayed) 2-cells is the unit type and the type of (displayed) 1-cells is a proposition.

Since the type of (displayed) morphisms (resp. objects) is a proposition (resp. a set), it remains to show that given a category equipped with a tensor, unit and left unitors λ_1, λ_2 (resp. right unitors and associators), then $\lambda_1 = \lambda_2$ under the assumption that the identity functor has a proof witnessing that the identity functor from $(\mathcal{C}, \otimes, I, \lambda_1)$ to $(\mathcal{C}, \otimes, I, \lambda_2)$ (and vice versa) preserves the left unitor. This is immediate. ◀

▶ **Lemma 28** (`assunitors_disp_locally_groupoidal`). $\mathbf{Cat}_{UA}$ *is locally groupoidal.*

Proof. This follows from the following lemmas:

1. The product of locally groupoidal displayed bicategories is locally groupoidal.
2. A displayed bicategory whose type of displayed 2-cells is the unit is locally groupoidal.

◀

Since a full displayed sub-bicategory of a univalent displayed bicategory is univalent, we conclude:

▶ **Lemma 29** (`tripent_disp_is_univalent_2`). $\mathbf{Cat}_P$ *is univalent.*

Since the full displayed sub-bicategory of a displayed locally groupoidal bicategory is locally groupoidal, we have that $\mathbf{Cat}_P$ is locally groupoidal.

▶ **Theorem 30** (`UMONCAT_is_univalent_2`). *The bicategory of univalent monoidal categories, lax monoidal functors, and monoidal natural transformations is univalent.*

▶ **Lemma 31** (`UMONCAT_disp_strong_is_univalent_2`). $\mathbf{Cat}_S$ *is univalent.*

Proof. This follows immediately from Theorem 30 since the type of displayed 1-cells is a mere proposition. ◀

▶ **Theorem 32** (`UMONCAT_strong_is_univalent_2`). *The bicategory of univalent monoidal categories, strong monoidal functors, and monoidal natural transformations is univalent.*

4 The Rezk Completion for Monoidal Categories

Some constructions of (monoidal) categories do not yield univalent (monoidal) categories. For instance, categories built from syntax usually have *sets* of objects; the presence of non-trivial isomorphisms in such a category hence entails that it is not univalent. Another example is when constructing colimits of univalent monoidal categories; the usual construction of such a colimit often yields a non-univalent monoidal category. In such cases, a "completion operation", turning a monoidal category into a univalent one, is handy.

In this section we construct, for each monoidal category, a free univalent monoidal category, which we call the *monoidal Rezk completion.* More precisely, we solve the following problem:

▶ **Problem 33.** *Given a Rezk completion $\mathcal{H} : \mathcal{C} \to \mathcal{D}$ of a category $\mathcal{C}$ and a monoidal structure $M := (\otimes, I, \lambda, \rho, \alpha)$ on $\mathcal{C}$, to construct a monoidal structure $\hat{M} := (\hat{\otimes}, \hat{I}, \hat{\lambda}, \hat{\rho}, \hat{\alpha})$ on $\mathcal{D}$ and a strong monoidal functor $\mathcal{H} : (\mathcal{C}, M) \to (\mathcal{D}, \hat{M})$ such that for any univalent monoidal category $(\mathcal{E}, N)$, the isomorphism of categories*

$$\mathcal{H} \cdot (-) : \mathbf{Cat}(\mathcal{D}, \mathcal{E}) \to \mathbf{Cat}(\mathcal{C}, \mathcal{E})$$

lifts to the category of lax (resp. strong) monoidal functors:

$$\mathcal{H} \cdot (-) : \mathbf{MonCat}((\mathcal{D}, \hat{M}), (\mathcal{E}, N)) \to \mathbf{MonCat}((\mathcal{C}, M), (\mathcal{E}, N)) \ .$$

Once solved, we call $(\mathcal{D}, \hat{M})$ the *monoidal Rezk completion of* $(\mathcal{C}, M)$. Analogous to the Rezk completion for categories, the monoidal Rezk completion exhibits the bicategory $\mathbf{MonCat}_{Univ}$ (resp. $\mathbf{MonCat}^{stg}_{Univ}$) as a reflective full sub-bicategory of $\mathbf{MonCat}$ (resp. $\mathbf{MonCat}^{stg}$).

Although any categorical structure on a category can be transported along an equivalence of categories such that they become equivalent in the corresponding bicategory of structured categories, this might not be the case if one considers a weak equivalence. On the way towards solving Problem 33, we show, in particular, how to transport a monoidal structure along a weak equivalence of categories, provided that the target category is univalent. That construction is not limited to the specific weak equivalence given by the Rezk completion.

Analogous to the univalence proof of $\mathbf{MonCat}_{Univ}$ (resp. $\mathbf{MonCat}^{stg}_{Univ}$) given in Section 3, we rely on the theory of displayed categories in order to solve this problem by dividing it into subgoals. In each of the subgoals, we use the same strategy. In Section 4.1, we explain the strategy in detail for the subgoal of equipping $\mathcal{D}$ (resp. $\mathcal{H} : \mathcal{C} \to \mathcal{D}$) with a tensor (resp. tensor-preserving structure).

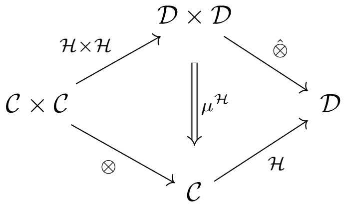
4.1 The Rezk Completion of a category with a tensor

Let $\mathcal{C}$ be a category and $\mathcal{H} : \mathcal{C} \to \mathcal{D}$ a Rezk completion of $\mathcal{C}$. Let $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ be a functor.

In this section we equip $\mathcal{D}$ with a functor $\hat{\otimes} : \mathcal{D} \times \mathcal{D} \to \mathcal{D}$ such that

1. $\mathcal{H}$ has the structure of a *strong tensor-preserving* functor, i. e., we have a natural isomorphism $\mu^{\mathcal{H}} : (\mathcal{H} \times \mathcal{H}) \cdot \hat{\otimes} \Rightarrow \otimes \cdot \mathcal{H}$.
2. The precomposition functor of $(\mathcal{H}, \mu^{\mathcal{H}})$ is an isomorphism of categories.

▶ **Definition 34** (`TransportedTensor`, `TransportedTensorComm`). *The* lifted tensor $\hat{\otimes}$ *on $\mathcal{D}$ is the lift of $\otimes \cdot \mathcal{H}$ along the weak equivalence $\mathcal{H} \times \mathcal{H} : \mathcal{C} \times \mathcal{C} \to \mathcal{D} \times \mathcal{D}$, i. e., $\hat{\otimes}$ is a functor* *together with a natural isomorphism:*



▶ Remark 35. The natural isomorphism is labelled as $\mu^{\mathcal{H}}$ because this natural isomorphism is precisely the structure we need to have that $\mathcal{H}$ is a (strong) tensor-preserving functor.

▶ **Lemma 36** (`HT_eso`)**.** *Let $\mathcal{E}$ be a univalent category and $\otimes_{\mathcal{E}} : \mathcal{E} \times \mathcal{E} \to \mathcal{E}$ be a functor. The displayed precomposition functor (Definition 21) $\mu^{\mathcal{H}} \cdot (-)$ with target displayed object $\otimes_{\mathcal{E}}$ is displayed split essentially surjective. Consequently, the precomposition functor $(\mathcal{H}, \mu^{\mathcal{H}}) \cdot (-)$ with target object $(\mathcal{E}, \otimes_{\mathcal{E}})$ between the tensor-preserving functor categories is merely essentially surjective on objects.*

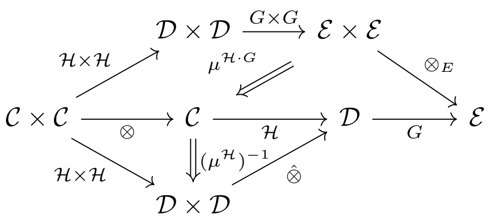
Proof. Let $G : \mathcal{D} \to \mathcal{E}$ be a functor and $\mu^{\mathcal{H}\cdot G}$ a natural transformation witnessing that $\mathcal{H} \cdot G$ is a lax tensor-preserving functor. We have to construct a natural transformation witnessing that G is a lax tensor-preserving functor, i. e., we have to define a natural transformation

$$\mu^G : (G \times G) \cdot \otimes_{\mathcal{E}} \Rightarrow \hat{\otimes} \cdot G \ .$$

Since $\mathcal{H} \times \mathcal{H}$ is a weak equivalence and $\mathcal{E}$ is univalent, it suffices to define a natural transformation of type

$$(\mathcal{H} \times \mathcal{H}) \cdot (G \times G) \cdot \otimes_{\mathcal{E}} \Rightarrow (\mathcal{H} \times \mathcal{H}) \cdot \hat{\otimes} \cdot G \ .$$

Thus we define μ^G as the lift of the natural transformation:



For a detailed proof that $\mu^{\mathcal{H}\cdot G}$ is (displayed) isomorphic to the (displayed) composition of $\mu^{\mathcal{H}}$ and μ^G, we refer the reader to the formalization. ◀

▶ **Lemma 37** (`HT_ff`)**.** *Let $\mathcal{E}$ be a univalent category and $\otimes_{\mathcal{E}} : \mathcal{E} \times \mathcal{E} \to \mathcal{E}$ be a functor. The displayed precomposition functor $\mu^{\mathcal{H}} \cdot (-)$ is displayed fully faithful. Consequently, the precomposition functor $(\mathcal{H}, \mu^{\mathcal{H}}) \cdot (-)$ between the tensor-preserving functor categories is fully faithful.*

Proof. It is displayed faithful because the type witnessing that a natural transformation preserves a tensor is a mere proposition. In order to show that it is displayed full, notice that we have to show an equality of morphisms, i. e., a proposition. Therefore, we are able to use that $\mathcal{H} \times \mathcal{H}$ is merely essentially surjective on objects which allows us to work with objects in $\mathcal{C}$ instead of $\mathcal{D}$ which leads to the result. ◀

▶ **Theorem 38** (precomp_tensor_catiso)**.** *A category equipped with a tensor admits a Rezk completion: Let $(\mathcal{E}, \otimes_{\mathcal{E}}) : \int \mathbf{Cat}_T$. If $\mathcal{E}$ is univalent, then*

$$(\mathcal{H}, \mu^{\mathcal{H}}) \cdot (-) : \int \mathbf{Cat}_T((\mathcal{D}, \hat{\otimes}), (\mathcal{E}, \otimes_E)) \to \int \mathbf{Cat}_T((\mathcal{C}, \otimes), (\mathcal{E}, \otimes_E))$$

is an isomorphism of categories.

Proof. First notice that both categories are univalent, indeed: since $\mathcal{E}$ is univalent, so are $\mathbf{Cat}(\mathcal{D}, \mathcal{E})$ and $\mathbf{Cat}(\mathcal{C}, \mathcal{E})$ and in Section 3, we have proven that the displayed bicategory $\mathbf{Cat}_T$ is locally univalent, i. e., the displayed hom-categories are univalent. So in order to show the result, it suffices to show that this functor is a weak equivalence, i. e., fully faithful and merely essentially surjective on objects. Fully faithfulness can always be concluded if both the functor on the base categories and the displayed functor are. However, in general it is not sufficient to conclude that a total functor is essentially surjective on objects if this holds on the base and at the displayed level. Fortunately, it does hold under the condition that the base category and displayed category of the codomain are univalent. So we conclude the result from combining the assumption that $\mathcal{H}$ is a weak equivalence and lemmas 37 and 36. ◀

▶ Remark 39. The strategy introduced in this section will be repeated in the next section, so we refer back to this section for the necessary details (if needed).

4.2 The Rezk Completion of a category with a tensor and unit

In Section 4.1, we have shown how the structure of a tensor $\otimes$ on $\mathcal{C}$ transports/lifts along a weak equivalence $\mathcal{H} : \mathcal{C} \to \mathcal{D}$ to a tensor on a univalent category $\mathcal{D}$. Furthermore, $\mathcal{H}$ has the structure of a strong monoidal functor and that $(\mathcal{D}, \hat{\otimes})$ is universal in a certain sense, i. e., objects in $\int \mathbf{Cat}_T$ admit a Rezk completion.

In this section, we show the same result holds when we add the choice of an object to a category, playing the role of the tensorial unit. This construction is trivial, but we will also discuss how we can conclude that objects in $\int \mathbf{Cat}_{TU}$ admit a Rezk completion.

As before, let $\mathcal{H} : \mathcal{C} \to \mathcal{D}$ be a weak equivalence from a category $\mathcal{C}$ to a univalent category $\mathcal{D}$. Let $I : \mathcal{C}$, thus $(\mathcal{C}, I) : \int \mathbf{Cat}_U$. Clearly we have $(\mathcal{H}, \mathsf{Id}_{\mathcal{H}\, I}) : \mathbf{Cat}_U((\mathcal{C}, I), (\mathcal{D}, \mathcal{H}\, I))$.

By the same reasoning as in Section 4.1, in order to conclude that $(\mathcal{D}, \mathcal{H}\, I)$ is universal, we have to show that for any $(\mathcal{E}, I_{\mathcal{E}}) : \mathbf{Cat}_U$ with $\mathcal{E}$ univalent, the displayed precomposition functor

$$\mathsf{Id}_{\mathcal{H}\, I} \cdot (-) : \mathbf{Cat}_U(\hat{I}, I_{\mathcal{E}}) \to \mathbf{Cat}_U(I, I_{\mathcal{E}})$$

is displayed fully faithful and displayed split merely essentially surjective on objects.

▶ **Lemma 40** (HU_eso)**.** *The displayed precomposition functor (Definition 21) $\epsilon^{\mathcal{H}} \cdot (-)$ with target displayed object $I_{\mathcal{E}}$ is displayed split essentially surjective. Consequently, the precomposition functor $(\mathcal{H}, \epsilon^{\mathcal{H}}) \cdot (-)$ with target object $(\mathcal{E}, I_{\mathcal{E}})$ between unit tensor-preserving functor categories is merely essentially surjective on objects.*

Proof. It is merely surjective since the witness, expressing that the weak equivalence preserves the unit, is an identity morphism. ◀

▶ **Lemma 41** (HU_ff)**.** *The displayed precomposition functor $\epsilon^{\mathcal{H}} \cdot (-)$ is displayed fully faithful. Consequently, the precomposition functor $(\mathcal{H}, \epsilon^{\mathcal{H}}) \cdot (-)$ between the unit-preserving functor categories is fully faithful.*

Proof. It is displayed faithful since the type of 2-cells is a property. It is displayed full as it follows immediately from the assumptions since the witness expressing that the weak equivalence preserves the unit is an identity morphism. ◀

Using the exact same reasoning used in Theorem 38, we conclude:

▶ **Theorem 42** (`precomp_unit_catiso`). *A category equipped with a unit admits a Rezk completion: Let* $(\mathcal{E}, I_{\mathcal{E}}) : \int \mathbf{Cat}_U$*. If* $\mathcal{E}$ *is univalent, then*

$$(\mathcal{H}, \epsilon^{\mathcal{H}}) \cdot (-) : \int \mathbf{Cat}_U((\mathcal{D}, \hat{I}), (\mathcal{E}, I_{\mathcal{E}})) \to \int \mathbf{Cat}_U((\mathcal{C}, I), (\mathcal{E}, I_{\mathcal{E}}))$$

is an isomorphism of categories.

So we have proven that objects in $\mathbf{Cat}_T$ and $\mathbf{Cat}_U$ admit a Rezk completion. From these results, we conclude that objects in $\mathbf{Cat}_{TU}$ admit a Rezk completion:

▶ **Theorem 43** (`precomp_tensorunit_catiso`). *Let* $(\mathcal{E}, \otimes_{\mathcal{E}}, I_{\mathcal{E}}) : \mathbf{Cat}_{TU}$*. If* $\mathcal{E}$ *is univalent, then*

$$(\mathcal{H}, \mu^{\mathcal{H}}, \epsilon^{\mathcal{H}}) \cdot (-) : \int \mathbf{Cat}_{TU}((\mathcal{D}, \hat{\otimes}, \hat{I}), (\mathcal{E}, \otimes_{\mathcal{E}}, I_{\mathcal{E}})) \to \int \mathbf{Cat}_{TU}((\mathcal{C}, \otimes, I), (\mathcal{E}, \otimes_{\mathcal{E}}, I_{\mathcal{E}}))$$

is an isomorphism of categories, i. e., objects in $\int \mathbf{Cat}_{TU}$ *admit a Rezk completion.*

Proof. Since the product of univalent displayed bicategories is again univalent, both the domain and codomain of this functor are univalent. Hence, by the same argument as in Theorem 38, it reduces to prove that the displayed precomposition functor is a displayed weak equivalence. The displayed precomposition functor is the product of the displayed precomposition functors of $\mu^{\mathcal{H}}$ resp. $\epsilon^{\mathcal{H}}$. Since the product of displayed weak equivalences is again a weak equivalence, the result now follows. ◀

4.3 The Rezk Completion of a category with a tensor, unit, unitors and associator

In this section, we prove that every object in $\int \mathbf{Cat}_{LU}$ (resp. $\mathbf{Cat}_{RU}$ and $\mathbf{Cat}_A$) has a Rezk completion.

As above, we let $\mathcal{H} : \mathcal{C} \to \mathcal{D}$ be a weak equivalence from a category $\mathcal{C}$ to a univalent category $\mathcal{D}$, $\mathcal{C}$ is equipped with a tensor $\otimes$ and a unit I. The lifted tensor on $\mathcal{D}$ is denoted by $\hat{\otimes}$ and $\hat{I} := \mathcal{H}\, I$. The witness that $\mathcal{H}$ preserves the tensor (resp. unit) (strongly) is denoted by $\mu^{\mathcal{H}}$ (resp. $\mu^{\mathcal{H}} = \mathsf{Id}_{\mathcal{H}\, I}$).

Before lifting a left unitor from $\mathcal{C}$ to $\mathcal{D}$, we first define a natural isomorphism stating that the weak equivalence preserves tensoring with the unit object (on the left):

▶ **Lemma 44** (`LiftPreservesPretensor`). *There is a natural isomorphism* $\mathcal{H} \cdot (\hat{I} \,\hat{\otimes}\, -) \Rightarrow (I \otimes -) \cdot \mathcal{H}$*.*

Proof. This is given by the following composition:

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\mathcal{H}} & \mathcal{D} \\ {\scriptstyle (I,-)}\downarrow & & \downarrow{\scriptstyle (\hat{I},-)} \\ \mathcal{C} \times \mathcal{C} & \xrightarrow{\mathcal{H} \times \mathcal{H}} & \mathcal{D} \times \mathcal{D} \\ {\scriptstyle \otimes}\downarrow & \mu^{\mathcal{H}} \Downarrow & \downarrow{\scriptstyle \hat{\otimes}} \\ \mathcal{C} & \xrightarrow[\mathcal{H}]{} & \mathcal{D} \end{array}$$

where the upper square is given by a trivial equality of functors. ◀

▶ **Definition 45** (`TransportedLeftUnitor`). *Let λ be a left unitor on $(\mathcal{C}, \otimes, I)$, i. e., $(\mathcal{C}, \otimes, I, \lambda) : \mathbf{Cat}_{LU}$. The* lifted left unitor *$\hat{\lambda}$ on $(\mathcal{D}, \hat{\otimes}, \hat{I})$ is the lift along $\mathcal{H}$ of the natural isomorphism given by the vertical composition of the natural isomorphism defined in Lemma 44 and $\lambda \triangleright \mathcal{H}$.*

An immediate calculation shows:

▶ **Lemma 46** (`H_plu`). *$\mathcal{H}$ preserves the left unitor.*

▶ **Theorem 47** (`precomp_lunitor_catiso`). *The objects in $\int \mathbf{Cat}_{LU}$ admit a Rezk completion:*

Let $(\mathcal{E}, \otimes_{\mathcal{E}}, I_{\mathcal{E}}, \lambda_{\mathcal{E}}) : \int \mathbf{Cat}_{LU}$. If $\mathcal{E}$ is univalent, then $(\mathcal{H}, \mu^{\mathcal{H}}, \epsilon^{\mathcal{H}}, \mathsf{plu}^{\mathcal{H}}) \cdot (-)$ of type

$$\int \mathbf{Cat}_{LU}((\mathcal{D}, \hat{\otimes}, \hat{I}, \hat{\lambda}), (\mathcal{E}, \otimes_{\mathcal{E}}, I_{\mathcal{E}}, \lambda_{\mathcal{E}})) \to \int \mathbf{Cat}_{LU}((\mathcal{C}, \otimes, I, \lambda), (\mathcal{E}, \otimes_{\mathcal{E}}, I_{\mathcal{E}}, \lambda_{\mathcal{E}}))$$

is an isomorphism of categories, where $\mathsf{plu}^{\mathcal{H}}$ is a witness that $\mathcal{H}$ preserves the left unitor (as provided by Lemma 46).

Proof. As before, it reduces to show that the displayed precomposition functor (Definition 21) is a displayed weak equivalence. It is displayed fully faithful since the type of 2-cells in $\mathbf{Cat}_{LU}$ is the unit type. We now show that it is displayed split essentially surjective on objects. Let $G : \mathcal{D} \to \mathcal{E}$ be a lax tensor and unit preserving functor such that $\mathcal{H} \cdot G$ preserves the left unitor. We have to show that G also preserves the left unitor. Since we have to show a proposition, the claim now follows from combining the essentially surjectiveness of $\mathcal{H}$ and then applying the assumption on $\mathcal{H} \cdot G$. ◀

Completely analogous is the case of right unitor:

▶ **Theorem 48** (`precomp_runitor_catiso`). *The objects in $\int \mathbf{Cat}_{RU}$ admit a Rezk completion.*

In order to prove that every object in $\int \mathbf{Cat}_{A}$ has a Rezk completion, we use an analogous trick as is used for objects in, e. g., $\int \mathbf{Cat}_{LU}$. An associator for $(\mathcal{D}, \hat{\otimes})$ is a natural isomorphism between functors of type $(\mathcal{D} \times \mathcal{D}) \times \mathcal{D} \to \mathcal{D}$. Since the product of weak equivalences is again a weak equivalence, such a natural isomorphism corresponds uniquely to a natural isomorphism between functors of type $(\mathcal{C} \times \mathcal{C}) \times \mathcal{C} \to \mathcal{D}$. As with the left unitor, the lifted natural isomorphism does not have the same type as the associator on $\mathcal{C}$. In the case of the left unitor, we only had to provide a natural isomorphism to match the domain, but for the associator, we furthermore need a natural isomorphism to match the codomain.

▶ **Theorem 49** (`precomp_associator_catiso`). *The objects in $\int \mathbf{Cat}_{A}$ admit a Rezk completion.*

4.4 The Rezk Completion of a monoidal category

In this section, we are able to conclude that the objects in $\mathbf{MonCat}$ and $\mathbf{MonCat}^{stg}$ admit a Rezk completion.

In the previous sections, we have lifted all the structure of a monoidal category to a weakly equivalent univalent category.

However, it still remains to show that the lifted structure $(\mathcal{D}, \hat{\otimes}, \hat{I}, \hat{\lambda}, \hat{\rho}, \hat{\alpha})$ satisfies the properties of a monoidal category if $(\mathcal{C}, \otimes, I, \lambda, \rho, \alpha)$ does.

▶ **Lemma 50** (`TransportedTriangleEq`, `TransportedPentagonEq`). *The lifted monoidal structure satisfies the pentagon and triangle equalities: If the triangle (resp. pentagon) equality holds for $(\mathcal{C}, \otimes, I, \lambda, \rho, \alpha)$, then it also holds for $(\mathcal{D}, \hat{\otimes}, \hat{I}, \hat{\lambda}, \hat{\rho}, \hat{\alpha})$.*

▶ **Theorem 51** (precomp_monoidal_catiso). *Any monoidal category admits a Rezk completion (considered in the bicategory of lax monoidal functors).*

Proof. By the same argument as in Theorem 43, we know that the precomposition functor w. r. t. $\int \mathbf{Cat}_{UA}$ is an isomorphism of categories since it holds for $\mathbf{Cat}_{LU}$, $\mathbf{Cat}_{RU}$ and $\mathbf{Cat}_{A}$.

Since the hom-categories in $\mathbf{Cat}_P$ are the terminal category, the displayed precomposition functor (Definition 21) w. r. t. $\mathbf{Cat}_P$ is clearly a weak equivalence which concludes the proof. ◀

Next, we prove that any monoidal category admits a Rezk completion in the bicategory of strong monoidal functors. Concretely, we show the following theorem:

▶ **Theorem 52** (precomp_strongmonoidal_catiso). *Let $\mathcal{C}$ be a monoidal category and $\mathcal{D}$ the Rezk completion of $\mathcal{C}$ as constructed in Theorem 51. If $\mathcal{E}$ is a univalent monoidal category, then*

$$\mathcal{H} \cdot (-) : \mathbf{MonCat}^{stg}(\mathcal{D}, \mathcal{E}) \to \mathbf{MonCat}^{stg}(\mathcal{C}, \mathcal{E})$$

is an isomorphism of categories.

Proof. First note that $\mathcal{H}$ is indeed strong monoidal by the definition of $\mu^{\mathcal{H}}$ and $\epsilon^{\mathcal{H}}$. Hence, the statement is well-defined.

As before, we have to conclude that the displayed precomposition functor (Definition 21) $((\mu^{\mathcal{H}})^{-1}, (\epsilon^{\mathcal{H}})^{-1}) \cdot (-)$ is fully faithful and split essentially surjective.

The displayed precomposition functor is fully faithful since every type of displayed 2-cells in $\mathbf{MonCat}^{stg}$ is the unit type.

The displayed precomposition functor is split essentially surjective on objects since the lift of a natural isomorphism is a natural isomorphism. ◀

5 Conclusion

We have studied (the bicategory of) monoidal categories in univalent foundations. First, we showed that the bicategory of univalent monoidal categories is univalent. Second, we constructed a Rezk completion for monoidal categories; specifically, we lifted the Rezk completion for categories to the monoidal structure. Our technique also works for lax and oplax monoidal categories, with minimal modifications. We have not presented this work here, but the `UniMath` code is available online.[2]

The second result provides a blueprint for constructing completion operations for "categories with structure". By "structure", we mean categorical structure such as functors and natural transformations. Here, the main challenge is to define a suitable notion of signature that allows us to specify structure on a category. Work on this topic will be reported elsewhere.

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