The Rezk Completion for Elementary Topoi

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Abstract

The development of category theory in univalent foundations and the formalization thereof is an active field of research. In univalent foundations, one can distinguish different flavours of categories. The most prominent of those is the notion of a univalent category, where identities and isomorphisms of objects coincide. One consequence hereof is that equivalences and identities coincide for univalent categories. In particular, structure on categories transfer along equivalences of univalent categories. A key aspect in the study of univalent categories is the Rezk completion, which allows us to construct univalent categories from non-univalent ones.

In this work, we present a modular framework for extending the Rezk completion from categories to categories with structure. We demonstrate the modularity of our framework by lifting the Rezk completion from categories to elementary topoi in manageable steps.

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1 Introduction

In this work, we continue the development of category theory in univalent foundations [17]. One of the central notions herein is that of a *univalent category*, which has the advantage that structures are invariant under equivalences and that structures defined via universal properties become unique up to identity. A key facet in the theory of univalent categories is the Rezk completion, which provides a construction to obtain univalent categories from non-univalent categories. We present a framework for extending the Rezk completion from categories to categories with structure.

Univalent foundations Univalent foundations are foundations in which structure and property are invariant under equivalence. The axiomatic system underpinning univalent foundations is dependent type theory, which provides the basis for various proof assistants and provides internal languages for categories.

One of the key features of UF is how equality of types is handled. The underlying type theory already gives equality. In UF, however, one assumes the so-called univalence axiom (UA), which states that identities (equalities) of types coincide with equivalences of types. The original semantics of UF is in the category of simplicial sets [10].

The univalence axiom indeed guarantees that structures are invariant under equivalence [5, 6]. Furthermore, UA implies a variety of extensionality principles, such as function extensionality, which implies that equality of algebraic structures coincides with isomorphism of algebraic structures, and is incompatible with uniqueness of identity proofs.

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Univalent categories In univalent foundations, one can distinguish different flavours of categories. Categories contain a type of objects and a dependent family of morphisms. Usually, however, one considers additional requirements due to non-trivial identity types. This leads to two notions of category: set-categories and univalent categories. While both correspond to categories in the simplicial set model, univalent categories arise naturally in univalent foundations.

Set-categories behave more like categories in the traditional (classical) sense. Univalent categories, however, are specific to univalent foundations, relying on the non-trivial identity types and the univalence axiom.

Univalent categories are categories for which identities of objects and isomorphisms of objects coincide [2]. The univalence condition on categories is often very desirable and is motivated by various examples. In the simplicial set model, the univalence condition corresponds to the completeness condition of Segal spaces. Furthermore, between univalent categories, the different notions of "sameness" all coincide. These notions of sameness are equivalences, isomorphisms, and identities. In particular, structures on univalent categories also transport along equivalences between them.

The univalence condition extends suitably to many categorical structures, such as bicategories [1], monoidal categories [23], and enriched categories [19]. In [5, 6], a univalence condition has been formulated for higher-categorical structures, generalizing in particular the aforementioned structures.

The Rezk completion Various constructions on categories, however, often produce non-univalent categories. For example, the construction of the Kleisli category via Kleisli morphisms [19] and the Cauchy completion when constructed via objects and idempotent morphisms [18] generally produce a non-univalent category, even if the category we start with is univalent. Furthermore, the tripos-to-topos construction, a fundamental construction in topos theory, generally produces a non-univalent category [9, 13].

In [2], Ahrens, Kapulkin, and Shulman, constructed for every category a univalent category which is equivalent, in a weak sense, to the original category. This construction is referred to as the Rezk completion.

Even though the Rezk completion provides a way to obtain a univalent category, for some constructions, you want the Rezk completion to have additional structure. For example, although one can apply the Rezk completion to the output of tripos-to-topos construction, one still needs to construct a topos structure on the Rezk completion for this to extend to a tripos-to-univalent-topos construction.

Goal In this work, we present a framework to extend the Rezk completion from categories to structured categories. The framework is designed to be modular and applicable to a variety of structures that a category can possess. To test our framework, we consider elementary topoi since they are categories with a variety of structures. While elementary topoi are our case study, our technique is general and can be used to lift Rezk completions to other classes of structured categories as well.

1.1 Contributions

The contributions of this paper are as follows:

1. In Section 3, we define displayed universal arrows (Definition 8), and we give a technique to construct displayed universal arrows over the Rezk completion (Proposition 10).

2. In Section 4, we use the technique of Section 3 to construct the Rezk completion of various classes of structured categories, among which are elementary topoi (Theorem 12).

In addition, the results in this paper are formalized using UniMath [21]. We recall the material necessary to understand this paper in Section 2.

1.2 Formalization

Most of the results presented here are formalized in the Rocq -library UniMath on univalent mathematics [16, 21]. The logic underlying the UniMath library is an intensional dependent type theory whose universes satisfy the univalence axiom. We do not rely on the axiom of choice nor the law of excluded middle. We also rely propositional truncation.

The formalization, and this paper, makes heavily use of the already existing material on (bi)category theory in the UniMath -library and the accompanied literature [2, 1]. The environments annotated with a 🐓 denote that the result is formalized. Upon clicking 🐓, the reader is directed towards the HTML documentation. The commit number from which the documentation is generated is 94b49c3.

2 Preliminaries

In this section, we give a brief introduction to univalent foundations and the development of category theory therein. A comprehensive introduction to univalent foundations can be found in e. g. , [17, 14].

2.1 Univalent Foundations

Univalent foundations is a version of dependent type theory where types are identified if they are equivalent. As a consequence, structure and property are invariant under equivalences, and all kind of mathematical structures are identified up to isomorphism.

Type theory in a nutshell We assume that the reader is familiar with the basics of dependent type theory, and for a complete reference we reader the reader to [12]. Here we recall the notation necessary for the remainder of the paper.

The basic building blocks of type theory are types and terms. Types are denoted by $A, B, \ldots$, and we write $a : A$ for a term a of type A.

In our type theory, we have various type formers. In particular, types constitutes a type, denoted $\mathcal{U}$, i.e., $A : \mathcal{U}$. To avoid paradoxes, there is actually a hierarchy of universes. Furthermore, functions from A to B are terms in the function type $A \to B$. If A is a type and $B : A \to \mathcal{U}$ a type family, we denote by $\prod_{a:A} B(a)$ and $\sum_{a:A} B(a)$ the type of dependent functions and dependent pairs respectively. Finally, given a type A and terms $a, b : A$, we denote the identity type as $a = b$ or $a =_A b$.

Univalence axiom The univalence axiom is a property for universes and characterizes their identity types. Let $\mathcal{U}$ be a universe and $A, B : \mathcal{U}$ terms. Then we can consider the identity type $A =_{\mathcal{U}} B$. While terms in $A =_{\mathcal{U}} B$ witness that A and B are *the same*, one can consider a weaker notion of sameness: equivalence.

An **equivalence** from A to B is a function with a left and right inverse and the type of equivalences is denoted $A \simeq B$. There is a function $\mathsf{idtoweq}_{A,B} : (A =_{\mathcal{U}} B) \to (A \simeq B)$ which sends the reflexivity path to the identity equivalence. The universe $\mathcal{U}$ satisfies the **univalence axiom** if $\mathsf{idtoweq}_{A,B}$ is an equivalence of types for every $A, B : \mathcal{U}$.

The univalence axiom implies a variety of extensionality principles, such as function extensionality which says that identities of functions correspond to pointwise identities [17].

Homotopy levels In our type theory, identity types can contain more than one element. Hence, types can be classified up to the complexity of their identity types. Let A be a type. Then A is **contractible** if A is equivalent to the unit type. The type A is a **proposition**, if $a = b$ is contractible for every $a, b : A$. If for all $a, b : A$ the type $a = b$ is a proposition, then A is a **set**.

The **propositional truncation** of a type A is the smallest proposition $\|A\|$ with a map from A to $\|A\|$. More precisely, for a type A, there is a proposition $\|A\|$ which is universal among all propositions. That is, if B is a proposition and there is a function $A \to B$, then we also have a function $\|A\| \to B$. This allows us to distinguish between chosen structure and structure that merely exists. More precisely, if $B : A \to \mathcal{U}$ is a type family, then we say that there exists an $a : A$ such that B holds if $\|\sum_{a:A} B(a)\|$ is inhabited.

2.2 Categories in Univalent Foundations

In traditional/set-based mathematics, a category $\mathcal{C}$ consists in particular of a set of objects $\mathcal{C}_0$ and for every two objects x and y in $\mathcal{C}_0$, a set of morphisms $x \to y$. Furthermore, a category is equipped with composition operation on the morphisms and identity morphisms which are associative and unital. There are multiple translation hereof into univalent foundations. A naive translation hereof into univalent foundations is to replace sets by types. Of course, one can replace sets by the (UF) sets, but even the category of sets and functions does not satisfy this criteria. Nonetheless, if the types have arbitrary homotopy levels, then one would need to empose higher coherence conditions.

To ensure that equalities of morphisms are propositions, each $x \to y$ is assumed to be a set. In [2], such categories are referred to as *pre*categories, but we will simply call them **categories**.

A **univalent category** is a category for which identities and isomorphisms between two objects coincide. More precisely, let $\mathcal{C}$ be a category and denote by $(x \cong y)$ the type of isomorphisms from x to y, for $x, y : \mathcal{C}_0$. By path induction, we have a function $\mathsf{idtoiso}_{x,y} : (x = y) \to (x \cong y)$. Then a category $\mathcal{C}$ is **univalent** if $\mathsf{idtoiso}_{x,y}$ is an equivalence of types, for every $x, y : \mathcal{C}_0$. In particular, this implies that every $x = y$ is a set.

Many examples of categories are univalent. Examples include the category Set of sets and functions, presheaf categories, and categories of algebraic structures, such as groups and rings.

▶ **Notation 1.** *Composition is written in diagrammatic order. That is, if $f : x \to y$ and $g : y \to z$ are morphisms, we denote by $f \cdot g$ their composite.*

2.3 The Rezk Completion for Categories

Even though many categories are univalent, a variety of constructions do not produce univalent categories. For example, there are two constructions of the Kleisli category, either via Kleisli morphisms or via free algebras. While the latter construction always produce a univalent category, the former does not [19]. An analogous observation holds for the Cauchy completion of a category [18]. Indeed, if one construct the Cauchy completion as a full subcategory of the presheaf categories, one obtains a univalent category. In particular, if one starts with a not-necessarily univalent category, then the presheaf construction gives a univalent category. However, if one constructs the Cauchy completion as objects in the initial category together with an idempotent morphism we do not have a univalent category, not even when the category started with is univalent.

The tripos-to-topos construction does, in general, not produce univalent categories [13, 9]. Furthermore, as opposed to the previous examples, there does not seem to be alternative equivalent construction hereof which does produce univalent topoi.

Nonetheless, there is a general construction to find a suitable univalent replacement for these categories, known as the **Rezk completion**. In this section, we recall the main theory of the Rezk completion. First, we recall the definition of the Rezk completion in Definition 2. Then we recall the universal property characterizing the Rezk completion in Proposition 3 and how this property can be described 2-categorically (Corollary 4). Lastly, in Remark 5 we zoom in on how the Rezk completion can be constructed and how Corollary 4 needs to be adapted to take into account the construction details. The Rezk completion provides a universal solution to making a category univalent [2]. In [2], it is proven that the universal property of the Rezk completion can be characterized in terms of weak equivalences.

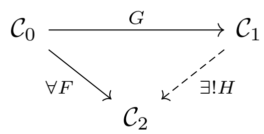
A functor $F : \mathcal{C}_1 \to \mathcal{C}_2$ is **essentially surjective** if for every object $y : \mathcal{C}_2$ in the codomain, there merely exists an object $x : \mathcal{C}_1$ in the domain, and an isomorphism of type $F\,x \cong y$. Furthermore, an essentially surjective functor F is a **weak equivalence** if F is fully faithful.

▶ **Definition 2** (Rezk completion). *A category $\mathcal{D}$ is a* ***Rezk completion*** *for $\mathcal{C}$ if $\mathcal{D}$ is univalent and there is a weak equivalence from $\mathcal{C}$ to $\mathcal{D}$.*

That the Rezk completion indeed satisfies the universal property of the free univalent completion follows more generally from the following proposition.

▶ **Proposition 3** (Thm 8.4. [2]). *Let $G : \mathcal{C}_0 \to \mathcal{C}_1$ be a weak equivalence between not-necessarily univalent categories. Then for every univalent category $\mathcal{C}_2$, the precomposition functor $(G \cdot -) : \mathsf{Cat}(\mathcal{C}_1, \mathcal{C}_2) \to \mathsf{Cat}(\mathcal{C}_0, \mathcal{C}_2)$ is an adjoint equivalence of univalent categories.*

In particular, this means that every functor into a univalent category can be extended along a weak equivalence, unique up to a natural isomorphism.



Furthermore, Rezk completions are unique up to an equivalence of categories. Due to univalence, Rezk completions are unique up to identity. Hence, for every category, its type of Rezk completions is a proposition. Thus we can say the Rezk completion.

To extend the Rezk completion to categories with structure, we consider (structured) categories as objects in a bicategory. One advantage hereof, is that we can treat structures uniformly and, in particular, we obtain modular constructions. Such an approach is indeed possible because the universal property can be phrased by saying that the inclusion of univalent categories into all categories has a left biadjoint.

Recall that a pseudofunctor $R : \mathcal{B}_1 \to \mathcal{B}_2$ has a **left biadjoint** L if we have a function $L : \mathcal{B}_2 \to \mathcal{B}_1$, a family of morphisms $\eta : \prod_{X:\mathcal{B}_2} X \to R(L(X))$ called **the unit**, and for every $X : \mathcal{B}_2$ and $Y : \mathcal{B}_1$, the functor $\eta_X \cdot R(-) : \mathcal{B}_1(L\,X, Y) \to \mathcal{B}_2(X, R\,Y)$ is an adjoint equivalence of categories.

Let Cat be the bicategory of categories, functors, and natural transformations. Denote by $\mathsf{Cat}_{\mathsf{univ}}$ the full subbicategory of Cat consisting of those categories that are univalent.

▶ **Corollary 4.** *Assume that for every category $\mathcal{C}$ a Rezk completion* $\mathsf{RC}(\mathcal{C})$ *is given, whose weak equivalence is denoted* $\eta_{\mathcal{C}} : \mathcal{C} \to \mathsf{RC}(\mathcal{C})$*. Then the inclusion of* $\mathsf{Cat}_{\mathsf{univ}}$ *into* Cat *has a left biadjoint* $\mathsf{RC} : \mathsf{Cat} \to \mathsf{Cat}_{\mathsf{univ}}$*. In particular, the action on objects is given by* $\mathcal{C} \mapsto \mathsf{RC}(\mathcal{C})$*, and the unit is pointwise given by* $\eta_{\mathcal{C}}$*.*

There are various constructions of the Rezk completion. However, there are some subtleties if we want to use Corollary 4 for them.

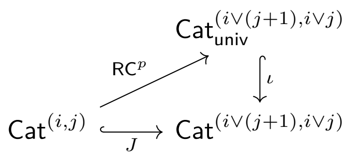
▶ Remark 5 (On the construction of $\mathsf{RC}(\mathcal{C})$). Ahrens, Kapulkin, and Shulman, showed that the Rezk completion of $\mathcal{C}$ can be constructed as a full subcategory of its category of presheaves $[\mathcal{C}^{\mathsf{op}}, \mathsf{Set}]$ [2]. More precisely, the Rezk completion be can be constructed as the replete full subcategory $\mathsf{RC}^p(\mathcal{C})$ of representable presheaves, and the weak equivalence is given by restricting the Yoneda embedding.

The *representable presheaf*-construction provides a general construction of the Rezk completion. Nonetheless, the category $\mathsf{RC}^p(\mathcal{C})$ lives in a higher universe level (see below). In concrete instances, one often has an alternative construction available that does not increase the universe level. For the Rezk completion, this can be done via higher inductive types [17].

For our purposes, lifting Rezk completions, the precise implementation of the Rezk completion is not relevant. Nonetheless, if we use the representable presheaf-construction, Corollary 4 has to be suitable adapted.

Recall that there is a hierarchy of universes and let $\mathcal{U}_k$ the universe at level k. Denote by $\mathsf{Cat}^{(i,j)}$ the type of categories whose type of objects is in $\mathcal{U}^{(i)}$ and where every $\mathcal{C}(x,y)$ is in $\mathcal{U}_j$. Then, for $\mathcal{C} : \mathsf{Cat}^{(i,j)}$ we have $\mathsf{RC}^p(\mathcal{C}) : \mathsf{Cat}^{(i \vee (j+1), i \vee j)}$, where $\vee$ is the least upperbound of universe levels. Hence, RC^p does *not* induce a pseudofunctor of type $\mathsf{Cat}^{(i,j)} \to \mathsf{Cat}^{(i,j)}_{\mathsf{univ}}$.

To take into account that the representable presheaf-construction raises the universe level, one can rephrase Corollary 4 in terms of relative left pseudoadjoints [7]. Instead, one can say that RC^p is a J-relative left pseudoadjoint to ι as depicted in following diagram:



3 On the Lifting of Biadjoints

Our goal is to lift the Rezk completion from categories to categories with additional structure, such as finitely complete categories. More precisely, we lift the left biadjoint to bicategories whose objects are structured categories and whose morphisms are structure preserving functors. In this section, we reduce the problem of lifting the Rezk completion in terms of weak equivalences, and we provide the general methodology we use.

The reduction step builds forth on the theory of displayed bicategories which provides a modular approach to the construction of bicategories and allows for a modular proof technique of e. g. , proving univalence for categories. We recall the necessary ingredients of this theory in Section 3.1 and we define the notion of *displayed universal arrows* in Definition 8 which gives a modular construction of (left) biadjoints, as witnessed by Proposition 9. In Section 3.2, we then apply Proposition 9 to obtain a formal description of the reduction step in Proposition 10.

3.1 Displayed Universal Arrows

In the remainder of this paper, we make heavy use of displayed bicategories, and we start by recalling this notion. Displayed bicategories serve various purposes, and one of these, is that we can use them to modularly construct bicategories and to modularly prove their univalence [1, 3]. To understand what displayed bicategories are, let us first recall an example of a displayed category. We can construct the category Mon of monoids and monoid homomorphisms by endowing the objects and morphisms of Set with extra structure and properties. Specifically, for every set there is a type of monoid structures on it, and for every function between sets with monoid structures on them we have a type expressing that this function is a homomorphisms. One must also prove that the identity function is a homomorphism, and that homomorphisms are closed under composition. Displayed categories generalize such descriptions of categories. Specifically, in a displayed category $\mathcal{D}$ over a category $\mathcal{C}$, we have a type $\mathcal{D}(x)$ of displayed objects over every object $x : \mathcal{C}$ and a set $\bar{x} \to_f \bar{y}$ of displayed morphisms for every morphism $f : x \to y$ and displayed objects $\bar{x} : \mathcal{D}(x)$ and $\bar{y} : \mathcal{D}(y)$. Displayed bicategories generalize displayed categories to the bicategorical setting, meaning that they also have displayed 2-cells with suitable compositions, associators, unitors, and coherences.

In Section 4 we consider various bicategories of structured categories. The objects of these bicategories are categories equipped a structure that is characterized by a universal property, the morphisms are functors which preserve this structure up to isomorphism. The 2-cells of these bicategories are all natural transformations. To construct such bicategories, we use a simplified version of displayed bicategories compared to [1]. Specifically, assume that the types of 2-cells are unit types, and we define them as follows.

▶ **Definition 6** (🐓). *Let $\mathcal{B}$ be a bicategory. A* ***displayed bicategory*** *with contractible 2-cells $\mathcal{D}$ over $\mathcal{B}$ consists of*

1. *for every $x : \mathcal{B}$, a type $\mathcal{D}(x)$; whose terms are called displayed objects;*
2. *for every $f : \mathcal{B}(x, y)$, $\overline{x} : \mathcal{D}(x)$, and $\overline{y} : \mathcal{D}(y)$, a type $\overline{x} \to_f \overline{y}$; whose terms are called displayed morphisms;*
3. *for every $x : \mathcal{B}$ and $\overline{x} : \mathcal{D}(x)$, a displayed morphism of type $\overline{x} \to_{\mathsf{id}_x} \overline{x}$;*
4. *for every $\overline{f} : \overline{x} \to_f \overline{y}$ and $\overline{g} : \overline{y} \to_g \overline{z}$, a displayed morphism of type $\overline{x} \to_{f \cdot g} \overline{z}$.*

Observe that no axioms are required since the axioms are phrased in terms of 2-cells.

The ***total bicategory*** *of $\mathcal{D}$, denoted $\int \mathcal{D}$, is the bicategory whose objects are (dependent) pairs $(x, \overline{x})$, with $x : \mathcal{B}$ and $\overline{x} : \mathcal{D}(x)$. The morphisms are pairs $(f, \overline{f})$ with $f : \mathcal{B}(\overline{x}, \overline{y})$ and $\overline{f} : \overline{x} \to_f \overline{y}$. The 2-cells are the 2-cells in $\mathcal{B}$.*

In the remainder of the paper, we assume that the types of 2-cells for each of our displayed bicategories is contractible. In particular, a displayed pseudofunctor between such displayed bicategories reduces to:

▶ **Definition 7** (🐓). *Let $F : \mathcal{B}_1 \to \mathcal{B}_2$ be a pseudofunctor and $\mathcal{D}_i$ a displayed bicategory over $\mathcal{B}_i$, for $i = 1, 2$. A* ***displayed pseudofunctor*** *over F consists of:*

1. *for every $x : \mathcal{B}_1$, a function $\hat{F} : \mathcal{D}_1(x) \to \mathcal{D}_2(F\,x)$;*
2. *for every $f : \mathcal{B}_1(x, y)$ and $\overline{x} : \mathcal{D}_1(x), \overline{y} : \mathcal{D}_1(y)$, a function $\hat{F} : (x \to_f y) \to (\hat{F}\,\overline{x} \to_{F\,f} \hat{F}\,\overline{y})$;*

The assignment $(x, \overline{x}) \to (F\,x, \hat{F}\,\overline{x})$ bundles into a pseudofunctor $\int_R \hat{R} : \int \mathcal{D}_1 \to \int \mathcal{D}_2$, and is referred to as the ***total pseudofunctor****.*

The following definition makes precise what is needed to lift a (left) biadjoint to displayed bicategories. For the definition of displayed adjoint equivalence between (1-)categories, we refer the reader to the formalization (also see pages 8-9 in [3], and [1] for such equivalences between displayed bicategories).

▶ **Definition 8** (). *Let R be a pseudofunctor from $\mathcal{B}_1$ to $\mathcal{B}_2$ with a left biadjoint (L, η). Let $\hat{R}$ be a displayed pseudofunctor from $\mathcal{D}_1$ to $\mathcal{D}_2$, over R. A* ***(left) displayed universal arrow*** *for $\hat{R}$ (over (R, L, η)) consists of:*

1. *$\hat{L} : \prod_{x:\mathcal{B}_2} \mathcal{D}_2(x) \to \mathcal{D}_1(L\,x)$, we write $\hat{L}(\overline{x}) := \hat{L}(x, \overline{x})$;*
2. *a family $\hat{\eta}_{\overline{x}} : \overline{x} \to_{\eta\,x} \hat{R}(\hat{L}(\overline{x}))$ of displayed morphisms, for all $(x, \overline{x})$ in $\int \mathcal{D}_2$;*

and such that the displayed functor between displayed hom-categories

$$\hat{\eta}_{\overline{x}} \cdot \hat{R}(-) : \mathcal{D}_1(\hat{L}\,\overline{x}, \overline{y}) \to \mathcal{D}_2(\overline{x}, \hat{R}\,\overline{y}),$$

is a displayed adjoint equivalence whose base of displayment is the adjoint equivalence $\eta_x \cdot R(-)$ given by (R, L, η).

▶ **Proposition 9** (). *Let $(\hat{R}, \hat{L}, \hat{\eta})$ be a displayed universal arrow for (R, L, η). Then the total pseudofunctor $\int_R \hat{R} : \int_{\mathcal{B}_1} \mathcal{D}_1 \to \int_{\mathcal{B}_2} \mathcal{D}_2$, has a left biadjoint.*

3.2 Displayed Universal Arrows over the Rezk Completion

In the remainder of this section, we apply Proposition 9 to reduce the lifting of the biadjunction in terms of weak equivalences and is made precise in Proposition 10.

Let $\mathcal{D}$ be a displayed bicategory over Cat. Then, $\mathcal{D}$ can be restricted to $\mathsf{Cat}_{\mathsf{univ}}$ by taking the (2-)pullback as depicted in the following diagram:

$$\begin{array}{ccc} \int \mathcal{D}_{\mathsf{univ}} & \xrightarrow{\iota} & \int \mathcal{D} \\ {\scriptstyle U}\downarrow \quad \lrcorner & & \downarrow{\scriptstyle U} \\ \mathsf{Cat}_{\mathsf{univ}} & \xrightarrow{\iota} & \mathsf{Cat} \end{array} \tag{1}$$

▶ **Proposition 10** (). *Let $\mathcal{D}$ be a displayed bicategory over* Cat *such that for every weak equivalence $G : \mathcal{C}_0 \to \mathcal{C}_1$, whose codomain is univalent, we have:*

1. *for $\overline{x} : \mathcal{D}(\mathcal{C}_0)$, there is a $\hat{x} : \mathcal{D}(\mathcal{C}_1)$ and a $\hat{G} : \overline{x} \to_G \hat{x}$;*
2. *for every univalent category $\mathcal{C}_2$, functors $F : \mathcal{C}_0 \to \mathcal{C}_2$ and $G : \mathcal{C}_1 \to \mathcal{C}_2$, and a natural isomorphism $\alpha : G \cdot H \Rightarrow F$, if $\overline{x}_i : \mathcal{D}(\mathcal{C}_i)$ and $\bar{F} : \overline{x}_0 \to_F \overline{x}_2$, then there is a $\bar{G} : \overline{x}_1 \to_G \overline{x}_2$.*

Then the pseudofunctor $\mathsf{RC} : \mathsf{Cat} \to \mathsf{Cat}_{\mathsf{univ}}$ *lifts to a left biadjoint for $\int \mathcal{D}_{\mathsf{univ}} \to \int \mathcal{D}$.*

▶ Remark 11 (). In Section 4, we stack displayed bicategories to obtain the displayed bicategory of elementary topoi. Proposition 10 works directly over Cat, and not over a total bicategory $\int_{\mathsf{Cat}} \mathcal{D}$. Nonetheless, if $\mathcal{E}$ is a displayed bicategory over $\int_{\mathsf{Cat}} \mathcal{D}$, the total bicategory $\int_{\int_{\mathsf{Cat}} \mathcal{D}} \mathcal{E}$ is equivalent to a displayed bicategory over Cat by applying the $\sum$ construction for displayed bicategories ([1, Definition 6.6]). In particular, if $\mathcal{D}$ and $\mathcal{D}'$ are both displayed bicategories over Cat, we can form their product, which is again a displayed bicategory over Cat.

4 The Rezk Completion Lifted To Topoi

In this section, we prove that the Rezk completion lifts from categories to categories equipped with structures defined via universal properties. In particular, we conclude that the Rezk completion for categories preserves many categorical structures. More precisely, we lift the biadjoint in Corollary 4 from categories to categories with the structures depicted in Figure 1, which allows us to conclude that the Rezk completions lifts from categories to topoi:

▶ **Theorem 12** (🏲🏲). *Let* $\mathsf{Top}_{\mathsf{el}}$ *be the bicategory of elementary topoi, logical functors, and natural transformations, and denote by* $(\mathsf{Top}_{\mathsf{el}})_{\mathsf{univ}}$ *its full subbicategory on elementary topoi whose underlying category are univalent. Then the inclusion* $(\mathsf{Top}_{\mathsf{el}})_{\mathsf{univ}} \hookrightarrow \mathsf{Top}_{\mathsf{el}}$ *has a left biadjoint* RC^{top}*. In particular:*

1. *the Rezk completion of an elementary topos* $\mathcal{E}$ *is an elementary topos* $\mathsf{RC}^{top}(\mathcal{E})$*;*
2. *the weak equivalence* $\eta_{\mathcal{E}} : \mathcal{E} \to \mathsf{RC}^{top}(\mathcal{E})$ *is a logical functor;*
3. *and,* $(\mathsf{RC}^{top}(\mathcal{E}), \eta_{\mathcal{E}})$ *is universal among the univalent elementary topoi.*

Furthermore, if $\mathcal{E}$ *has a parameterized natural numbers object (NNO) then* $\mathsf{RC}^{top}(\mathcal{E})$ *has an NNO,* $\eta_{\mathcal{E}}$ *preserves the NNO, and universality lifts to elementary topoi with NNOs.*

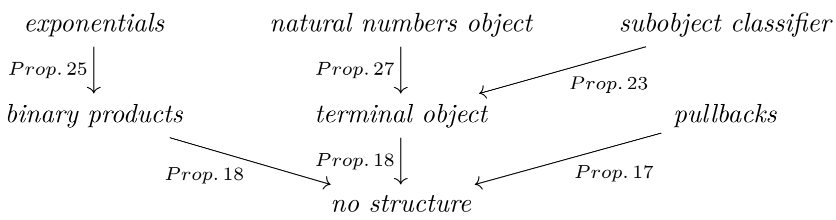


Figure 1 Tower Of Structures

(a) The labels of the vertices refer to the result stating the lifting.

To prove that the Rezk completion and the left biadjoint lift from categories to each of the structures we apply Proposition 10. That is, we prove that the two assumptions in Proposition 10 are satisfied. To do this, we take the following approach:

1. First, we show that given a weak equivalence, with possibly the assumption of univalence on the codomain, a structure on the domain transports onto the image of the codomain. From this, condition 1 is a direct consequence due to the properties of weak equivalences and possibly the univalence requirement which guarantees that the type witnessing the structure on the codomain is a proposition.
2. Second, to conclude condition 2, we first prove that weak equivalences reflect the structure. Then we apply essential surjectivity to work directly in the image of the weak equivalence and hence in the domain of the weak equivalence.

In Section 4.1, we spell out the approach and most of the details to conclude the Rezk completion for categories with pullbacks. The proof for the other structures follow the same approach.

For readability, we use the following notation:

▶ **Notation 13.** *The concrete bicategories we consider are constructed as displayed bicategories over* Cat*. For readability, we make no distinction between the bicategories and the displayed bicategories. For* $\mathcal{K} \in \{\mathcal{B}, \mathcal{D}\}$*, we write* $\mathcal{K}_{\mathsf{univ}}$ *to denote the restriction of* $\mathcal{K}$ *to univalent categories.*

4.1 (Co)Limits

In this subsection, we show that the Rezk completion for categories lifts to finitely complete categories. It is well-known that the existence of a terminal object and pullbacks implies the existence of all finite limits. Hence, it is sufficient to lift the Rezk completion for those limits. We only discuss the lifting of pullbacks and refer the reader to the formalization for the result on the other limits.

First, we show in Lemma 14 that the image of a pullback square under a weak equivalence is again a pullback square. From this, we conclude in Lemma 15 that the Rezk completion of a category with pullbacks also has pullbacks and that the weak equivalence into the Rezk completion (i.e., the unit) preserves pullbacks. Then, we show in Lemma 16 that weak equivalences reflect pullbacks which allows us to conclude that the biadjunction given by the Rezk completion lifts from categories to categories with pullbacks. Combined with the analogous results for the terminal object, we conclude in Proposition 18 that the biadjunction lifts from categories to finitely complete categories.

▶ **Lemma 14.** *Let $G : \mathcal{C}_0 \to \mathcal{C}_1$ be a weak equivalence. Then the image of a pullback square under G is again a pullback square.*

Proof. Assume that the following diagram is a pullback square in $\mathcal{C}_0$:

$$\begin{array}{ccc} p & \xrightarrow{\pi_2} & x_2 \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle p_2} \\ x_1 & \xrightarrow[p_1]{} & y \end{array}$$

We have to show that its image under G is again a pullback square. Hence, fix $\bar{z} : \mathcal{C}_1$, $\bar{f}_1 : \bar{z} \to G(x_1)$, $\bar{f}_2 : \bar{z} \to G(x_2)$ and assume $\bar{f}_1 \cdot F(p_1) = \bar{f}_2 \cdot F(p_2)$. Observe that

$$\exists!\, \bar{k} : \bar{z} \to F(p), \bar{k} \cdot F(\pi_1) = \bar{f}_1 \times \bar{k} \cdot F(\pi_2) = \bar{f}_2, \tag{2}$$

is a proposition. Thus, we can apply essential surjectiveness of G to get an object $z : \mathcal{C}$ and an isomorphism $i : G(z) \cong \bar{z}$. The result now follows from the usual proof that an equivalence of categories preserves limits. Indeed, let $f_1 := G^{-1}(i \cdot \bar{f}_1)$ and $f_2 := G^{-1}(i \cdot \bar{f}_2)$ where G^{-1} is the inverse function of the action of G on morphisms given fully faithfulness of G. Then, one can prove that (z, f_1, f_2) is a cone, where the commutativity of the diagram follows from the fact that a fully faithful functor reflects equality of morphisms. Hence, there exists a unique $k : z \to p$ such that $k \cdot \pi_1 = f_1$ and $k \cdot \pi_2 = f_2$. We can now show that $\bar{k} := i^{-1} \cdot G(k)$ is a proof of Equation (2) and, that $\bar{k}$ is necessarily unique. ◀

▶ **Lemma 15.** *Let $\mathcal{C}$ be a category equipped with pullbacks. Then $\mathsf{RC}(\mathcal{C})$ is equipped with pullbacks. Furthermore, $\eta_{\mathcal{C}} : \mathcal{C} \to \mathsf{RC}(\mathcal{C})$ preserves those pullbacks.*

Proof. Let $\bar{y}, \bar{x}_1, \bar{x}_2 : \mathsf{RC}(\mathcal{C})$ and $\bar{p}_1 : \mathsf{RC}(\mathcal{C})(\bar{x}_1, \bar{y})$ and $\bar{p}_2 : \mathsf{RC}(\mathcal{C})(\bar{x}_2, \bar{y})$. We have to construct the pullback of $\bar{p}_1$ along $\bar{p}_2$. Since $\mathsf{RC}(\mathcal{C})$ is univalent, the type of pullbacks of $\bar{p}_1$ along $\bar{p}_2$ is a proposition. Hence, by essential surjectiveness there are isomorphisms of types $\eta_{\mathcal{C}}(y) \cong \bar{y}$, $\eta_{\mathcal{C}}(x_1) \cong \bar{x_1}$, and $\eta_{\mathcal{C}}(x_2) \cong \bar{x_2}$, for some $y, x_1, x_2 : \mathcal{C}$. By univalence of $\mathsf{RC}(\mathcal{C})$, these isomorphisms correspond to identities. Hence, by induction on those identities, $\bar{p}_i$ is equivalently a morphism of type $\mathsf{RC}(\mathcal{C})(\eta_{\mathcal{C}}\, x_i, \eta_{\mathcal{C}}\, y)$ (for $i = 1, 2$). Thus, by fully faithfulness, $\bar{p}_i = \eta(p_i)$ for some $p_i : \mathcal{C}(x_i, y)$. That $\mathsf{RC}(\mathcal{C})$ has pullbacks now follows because we can take the pullback of p_1 and p_2 (in $\mathcal{C}$) and then use that weak equivalences preserve pullbacks (Lemma 14). ◀

▶ **Lemma 16.** *Every weak equivalence reflect pullbacks.*

Let $\mathsf{Cat}_{\mathsf{pb}}$ be the bicategory whose objects are categories equipped with pullbacks, whose morphisms are pullback preserving functors, and whose 2-cells are natural transformations.

▶ **Proposition 17.** *The inclusion $(\mathsf{Cat}_{\mathsf{pb}})_{\mathsf{univ}} \to \mathsf{Cat}_{\mathsf{pb}}$ has a left biadjoint.*

Proof. The first assumption of Proposition 10 has been proven in Lemma 14 and Lemma 15. Hence, it remains to prove that for every diagram

$$\begin{array}{ccc} \mathcal{C}_0 & \xrightarrow{G} & \mathcal{C}_1 \\ & \overset{\alpha}{\Leftrightarrow} & \\ {\scriptstyle F}\searrow & & \swarrow{\scriptstyle H} \\ & \mathcal{C}_2 & \end{array} \tag{3}$$

where G is a weak equivalence, we have if F preserves pullbacks, then so does H. Assume that the following diagram on the left is a pullback square in $\mathcal{C}_1$:

$$\begin{array}{ccc} p' & \xrightarrow{\pi'_2} & x'_2 \\ {\scriptstyle \pi'_1}\downarrow & & \downarrow{\scriptstyle p'_2} \\ x'_1 & \xrightarrow[p'_1]{} & y' \end{array} \qquad \begin{array}{ccc} H(p') & \xrightarrow{H(\pi'_2)} & H(x'_2) \\ {\scriptstyle H(\pi'_1)}\downarrow & & \downarrow{\scriptstyle H(p'_2)} \\ H(x'_1) & \xrightarrow[H(p'_1)]{} & H(y') \end{array} \qquad \begin{array}{ccc} F(p) & \xrightarrow{F(\pi_2)} & F(x_2) \\ {\scriptstyle F(\pi_1)}\downarrow & & \downarrow{\scriptstyle F(p_2)} \\ F(x_1) & \xrightarrow[F(p_1)]{} & F(y) \end{array} \tag{4}$$

We have to show that the diagram on the middle is a pullback. Since the type expressing that a square is a pullback is a proposition, essential surjectiveness implies that p', x'_1, x'_2, y' are isomorphic to the image of G of some objects $p, x_1, x_2, y : \mathcal{C}_0$. Furhermore, by fully faithfulness of G, the p'_i's and π'_i's correspond uniquely with morphisms in $\mathcal{C}_0$. More precisely, we define $\pi_i := G^{-1}\left(j_p \cdot \pi'_i \cdot j_{x_i}^{-1}\right) : \mathcal{C}_0(p, x_i)$ and $p_i := G^{-1}\left(j_{x_i} \cdot p'_i \cdot j_y^{-1}\right) : \mathcal{C}_0(x_i, y)$, where the j's are the isomorphisms given by essential surjectiveness. Since (π'_1, π'_2) is a pullback of (p'_1, p'_2), and since weak equivalences reflect pullbacks, (π_1, π_2) is a pullback of (p_1, p_2). Now, since F preserves pullbacks, we have that the right diagram in 4 is a pullback square. The claim now follows since the middle and right diagram are equivalent by α. ◀

Let FinLim be the bicategory whose objects are categories equipped with *a terminal object and pullbacks*, whose morphisms are functors preserving those limits, and whose 2-cells are natural transformations. This bicategory is defined via the product for displayed bicategories.

▶ **Proposition 18** (🐓). *The inclusion* $\mathsf{FinLim}_{\mathsf{univ}} \to \mathsf{FinLim}$ *has a left biadjoint.*

We define the bicategory $\mathsf{FinColim}$ of finitely cocomplete categories similarly. That is, as the product of displayed bicategories encoding binary coproducts, and coequalizers:

▶ **Corollary 19** (🐓). *The inclusion of* $\mathsf{FinColim}_{\mathsf{univ}} \to \mathsf{FinColim}$ *has a left biadjoint.*

Proof. This follows from the duality between limits and colimits and the fact that the opposite of weak equivalence is a weak equivalence between the opposite categories. ◀

▶ Remark 20 (Infinite (co)limits). Whereas (strong) equivalences of categories create arbitrary (co)limits, there is no reason for weak equivalences to create infinitary (co)limits, even if one assume the codomain to be univalent. Indeed, let $\eta_{\mathcal{C}} : \mathcal{C} \to \mathsf{RC}(\mathcal{C})$ be the Rezk completion and assume $(x_i)_{i:I}$ is a product cone in $\mathsf{RC}(\mathcal{C})$. Since the type of products for $(x_i)_i$ is a proposition, essential surjectivity could be used to an individual x_i, or a finite set, but not all simultaneously.

4.2 Subobject Classifier

In this subsection, we show that the Rezk completion for finitely complete categories lifts to such categories with a subobject classifier. We take the same steps as before. That is, we first show in Lemma 22 that the image of a subobject classifier is again a subobject classifier and then we show in Proposition 23 that the factorization of a subobject classifier preserving functor along a weak equivalence again preserves subobject classifiers. Both proofs also rely on the interaction between monomorphisms and weak equivalences (Lemma 21).

Let $\mathcal{C}$ be a category with a terminal object T. Recall that a **subobject classifier** consists of an object $\Omega : \mathcal{C}$ and a monomorphism $\tau : \mathcal{C}(T, \Omega)$, which furthermore satisfies the following universal property. For every mono $f : \mathcal{C}(x, y)$, there is a unique morphism $\chi_f : \mathcal{C}(y, \Omega)$ such that the following diagram commutes and is a pullback square:

$$\begin{array}{ccc} x & \xrightarrow{!} & T \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \tau} \\ y & \xrightarrow[\chi_f]{} & \Omega \end{array}$$

We refer to τ as the *truth map*.

Recall that a morphism $f : \mathcal{C}(x, y)$ is a monomorphism if and only if

$$\begin{array}{ccc} x & = & x \\ \| & & \downarrow{\scriptstyle f} \\ x & \xrightarrow[f]{} & y \end{array}$$

is a pullback square.

▶ **Lemma 21.** *Weak equivalences preserve and reflect monomorphisms.*

Recall that if G is a weak equivalence and T_0 a terminal object, then $G(T_0)$ is terminal. Lemma 22 follows since weak equivalences preserve pullbacks:

▶ **Lemma 22.** *Let $G : \mathcal{C}_0 \to \mathcal{C}_1$ be a weak equivalence between categories with a terminal object, denoted T_0 and T_1 respectively. Let $!$ be the unique morphism from T_1 to $F(T_0)$. If $(\Omega, \tau : T_0 \to \Omega)$ is a subobject classifier, then so is $F(\Omega)$ whose truth map is $! \cdot F(\tau) : T_1 \to F(\Omega)$.*

Recall that a functor $F : \mathcal{C}_0 \to \mathcal{C}_1$ between categories with a terminal object, denoted T_i for $i = 0, 1$, and a subobject classifier (Ω_i, τ_i) preserves the subobject classifier if one of the following two equivalent conditions hold:

1. $\left(F(\Omega_0), T_1 \xrightarrow{!} F(T_0) \xrightarrow{F(\tau_0)} F(\Omega_0) \right)$ is a subobject classifier in $\mathcal{C}_1$;
2. there is an isomorphism $i : F(\Omega_0) \simeq \Omega_1$ such that $F(\tau_0) \cdot i = !^{-1} \cdot \tau_1$.

Let FinLim_Ω be the bicategory whose objects are finitely complete categories equipped with a subobject classifier and whose morphisms are functors that preserve finite limits and the subobject classifier.

▶ **Proposition 23.** *The inclusion $(\mathsf{FinLim}_\Omega)_{\mathsf{univ}} \to \mathsf{FinLim}_\Omega$ has a left biadjoint.*

Proof. Again, it suffices to verify the conditions given in Proposition 10. The first condition is an immediate consequence of Lemma 22. The second condition follows because subobject classifiers are unique up to isomorphism and since weak equivalences reflect subobject classifiers. ◀

4.3 Cartesian Closedness

Now, we lift the Rezk completion from categories to cartesian closed categories, which concludes the first part in Theorem 12. We already know that the Rezk completion extends to finite products. Hence, it remains to verify that the Rezk completion extends to exponentials. Again, we apply the same approach as in the previous subsections.

Recall that an object $x : \mathcal{C}$ in a category with binary products is **exponentiable** if $- \times x : \mathcal{C} \to \mathcal{C}$ has a right adjoint, which is denoted as $(-)^x : \mathcal{C} \to \mathcal{C}$. That is, for every $y : \mathcal{C}$ there are given an object $y^x : \mathcal{C}$ and a morphism $\mathsf{ev} := \mathsf{ev}_{x,y} : \mathcal{C}(y^x \times x, y)$ which is universal in the sense that (y^x, ev) is the terminal such pair. The object y^x is referred to as the *exponential* of x with y, and $\mathsf{ev}_{x,y}$ is referred to as the *evaluation morphism*. We say that a category is **cartesian closed** if it has binary products and all exponentials.

Recall that the Rezk completion lifts to categories with binary products. In particular, there is a necessarily unique isomorphism $\mu := \mu_{x,y} : \eta(x) \times \eta(y) \cong \eta(x \times y)$, for every $x, y : \mathcal{C}$ and natural in x and y.

▶ **Lemma 24** (). *Let $\mathcal{C}_0$ be a cartesian closed category and $G : \mathcal{C}_0 \to \mathcal{C}_1$ a weak equivalence. Then, the image of an exponential, under G is again an exponential. That is, if (y^x, ev) is an exponential of x with y, then $(G(y^x), G(\mathsf{ev}))$ is an exponential of $G(x)$ with $G(y)$.*

A **cartesian closed functor** is a binary product preserving functor $F : \mathcal{C}_0 \to \mathcal{C}_1$ between cartesian closed categories which **preserves exponentials**. That is, for every $x, y : \mathcal{C}_0$ the unique morphism from $F(y^x)$ to $F(y)^{F(x)}$ is an isomorphism. The bicategory of cartesian closed categories, cartesian closed functors, and natural transformations is denoted CCC. Since exponentials are unique up to isomorphism, the preservation of exponentials is equivalent to the statement that the image of an exponential object is again an exponential object as in Lemma 24. Hence, Lemma 24 implies the first two conditions in Proposition 10 applied to $\mathcal{D} := \mathsf{CCC}$.

▶ **Proposition 25** (). *The inclusion $\mathsf{CCC}_{\mathsf{univ}} \to \mathsf{CCC}$ has a left biadjoint.*

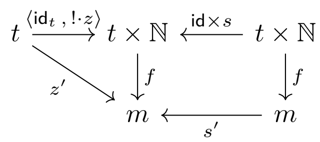
Hence, by combining Proposition 18, Proposition 23, and Proposition 25, we lifted the Rezk completion from categories to elementary topoi, which concludes the first part in Theorem 12.

4.4 Paramaterized NNO

We now prove the furthermore clause in Theorem 12. That is, the Rezk completion preserves (parameterized) natural numbers objects.

The most common way to interpret, or axiomatize, the object of natural numbers in a category leads to the definition of a *natural numbers object*. Nonetheless, in the absence of exponentials, there is a more appropriate interpretation by weakening the recursion principle and is known as a *parameterized natural numbers object* [11]. We work with the more general version.

A **parameterized natural numbers object** (NNO) in a category $\mathcal{C}$ with finite products is a tuple $(\mathbb{N}, z, s)$ where $\mathbb{N} : \mathcal{C}$ is an object and $z : \mathcal{C}(T, \mathbb{N})$, $s : \mathcal{C}(\mathbb{N}, \mathbb{N})$ are morphisms which satisfies the following universal property: for every tuple $(t : \mathcal{C}, m : \mathcal{C}, z' : \mathcal{C}(t, m), s' : \mathcal{C}(m, m))$, there exists a unique $f : \mathcal{C}(t \times \mathbb{N}, m)$ such that the following diagram commutes:



Recall that the image of a terminal object under a weak equivalence is again terminal.

▶ **Lemma 26** (). *Let $G : \mathcal{C}_0 \to \mathcal{C}_1$ be a weak equivalence where both $\mathcal{C}_0$ and $\mathcal{C}_1$ have binary products. Assume that $\mathbb{N} : \mathcal{C}_0$, $z : \mathcal{C}_0(T_0, \mathbb{N})$, and $s : \mathcal{C}_0(\mathbb{N}, \mathbb{N})$ are given. Then $(\mathbb{N}, z, s)$ is an NNO if and only if $(G(\mathbb{N}), G(z), G(s))$ is an NNO.*

Let $\mathsf{Cat}_{\mathbb{N}}$ be the bicategory whose objects are cartesian categories equipped with a parameterized NNO, and whose morphisms are terminal preserving functors $F : \mathcal{C}_0 \to \mathcal{C}_1$ such that the unique morphism from $\mathbb{N}_1 \to F(\mathbb{N}_0)$, induced by the universal property of the NNO, is an isomorphism.

▶ **Proposition 27** (). *The inclusion $(\mathsf{Cat}_{\mathbb{N}})_{\mathsf{univ}} \to \mathsf{Cat}_{\mathbb{N}}$ has a left biadjoint.*

▶ **Corollary 28** (). *The left biadjoint* RC *lifts to elementary topoi with NNO's.*

5 Related Work

We elaborate on the different methods to extend the Rezk completion to categories with structure found in the literature.

In [4], the Rezk completion has been extended to, in particular, categories with families (CwFs). That is, they showed that for weak equivalence into a univalent category (i.e., the Rezk completion), induces a map from the type of CwFs on the domain to the type of CwFs on the codomain. Furthermore, they showed that for every weak equivalence one has an equivalence between the representable map structures on the involved categories. These results allowed them to prove that there is an equivalence between representable maps of presheaves on a category and the CwFs on its Rezk completion. While our focus has been on displayed bicategories with contractible 2-cells, CwFs and representable maps require propositional 2-cells. Nonetheless, Proposition 10 could be slightly generalized to have propositional 2-cells. In particular, we expect our methodology to also work for these structures.

In [20], it was proven that the inclusion of the bicategory of univalent groupoids into the bicategory all groupoids admits a left biadjoint. Furthermore, it was shown that this biadjoint lifts to a variety of structures on groupoids. However, instead of considering each structure individually, as we have done, [20] presented a signature for higher inductive types to encode structure on groupoids and then showed that the structure definable by such HITs induces a left biadjoint for the inclusion of structured univalent groupoids into structured groupoids. In particular, [20] considers *algebraic* structure where as we consider *universal* structure on categories. Furthermore, while we work with biadjoints in terms of equivalences between hom-categories, [20] working with biadjoints in terms of the unit-counit description.

In [23], the Rezk completion has been extended to *monoidal categories*. In particular, they also proved that the biadjoint given by the Rezk completion lifts from categories to monoidal categories, although not in the exact same words. The approach considered here differs in the following way. First, we do not rely on the universal property of weak equivalences and the Rezk completion. Instead, we show the lifting in terms of weak equivalences. Second, whereas in [23], they relied on displayed categories to lift the equivalence on hom-categories, we now rely on displayed bicategories to formulate displayed biadjunctions/universal arrows.

Instead of working with *abstract Rezk completions*, one can also try to use a *concrete implementation* of the Rezk completion, either via the presheaf construction, or via higher inductive types. In [23], it was furthermore sketched how the presheaf construction inherits a monoidal structure by considering the Day convolution structure on the category of presheaves. In [19], both implementations of the Rezk completion have been extended to the setting of enriched categories. However, as opposed to the monoidal case, some modifications have been done. First, the notion of fully faithfulness, and hence weak equivalence, has to be adapted to take the enrichment into account. Nonetheless, [19] showed that such weak equivalences imply an equivalence on the *enriched functor categories*, similar universal to the (non-enriched) weak equivalences. Second, as opposed to the monoidal Rezk completion, [19] considered enriched presheaves, as opposed to Set-valued presheaves. Furthermore, while the presheaf construction has been adapted, the HIT construction for the enriched Rezk completion shows that the underlying category of the enriched Rezk completion is the *ordinary* Rezk completion.

6 Conclusion

We have presented a modular framework for the lifting of Rezk completions from categories to categories with structures and concluded that the Rezk completion of an elementary topos is again an elementary topos. In particular, our main result implies that the interpretation of type-formers and logical constructs is preserved under the Rezk completion.

There are multiple ways in which this work can be extended. First, in [1], it is proven that the presheaf construction for the Rezk completion of (1-)categories induces a similar result when passing to (locally univalent) bicategories. Hence, there is the question of how the results presented here *categorify* to higher categories and topoi (see e. g. , [15, 22]).

Second, we can compose the tripos-to-topos construction with the Rezk completion for topoi, which provides a *tripos-to-univalent-topos construction*. Nonetheless, if one uses this construction to get realizability topoi, it remains to be seen whether the obtained topoi share the same structures and properties as realizability topoi. Hence, one would need to verify the properties of the Giraud-like theorem for realizability topoi [8].

Third, there are more categorical structures one can consider. One such example is *locally cartesian closedness*, used to interpret Π-types. We expect this to follow from the results about cartesian closed categories presented above and the fact that taking slice categories is well-behaved with respect to weak equivalences. One can also consider examples outside of topos theory. For example, abelian categories can be characterized in an *enrichment-free* way using (finitary) universal constructions, whose preservation properties should follow from the above.

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