Universiteit Antwerpen

Masterproef scriptie

Two-categorical Giraud theorems

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Abstract

In this thesis we study how the theory of Grothendieck topoi is generalized to the setting of two-categories. The goal of this thesis to show a two-dimensional version of the theorem of Giraud (which characterizes grothendieck topoi as certain exact categories) and other well-known characterizations.

Acknowledgments

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Chapter 0

Introduction

Sheaves are a central object in the study of Algebraic Geometry, it is therefore natural to consider the category of sheaves (over a fixed topological space, or more generally a site). Categories which are equivalent to such a category of sheaves are called a *Grothendieck topoi*. A well-known theorem in topos theory, called *Giraud's Theorem* characterizes such categories without mentioning the underlying geometric structure. In the paper [10], Ross Street introduces how topos theory can be generalized to 2-categories, i.e. categories whose hom-sets are again categories. Moreover, he generalizes Giraud's theorem to the setting of 2-categories.

In the first chapter we introduce the most important concepts of the theory of 2-categories, like the Yoneda lemma, (weighted) limits and we also introduce the notion of acute and chronic arrows which play the role of epi -and monomorphisms in 2-categories. These are needed to prove the two-dimensional theorem of Giraud.

Recall that the theorem of Giraud characterizes Grothendieck topoi as certain exact categories. A fundamental notion of an exact category is that of an equivalence relation. This concept is generalized to 2-categories by R. Street and is called a *congruence*. In chapter two we introduce this notion. This is defined as a internal functor, hence we first introduce the definitions of an internal category -and functor. The archetypal example of a 2-category is a the category **Cat** of small categories, so we look at the internal categories in **Cat** and we show that these are precisely the *double categories*.

In the final chapter, we study the theory of topoi in a 2-categorical context. This is done similarly to the theory of topoi in a one-dimensional context. Namely, by first defining sheaves, how the sheaf property can be characterized as a limit (just as the ordinary sheaf property is equivalent to some equalizer property). We then consider the sheafification which allows us to characterize 2-topoi as localizations of presheaf-categories and then we work up to the 2-dimensional theorem of Giraud which we call *Street's theorem*.

Chapter 1

Two categories

1.1 Introduction two-categories

By **Cat**, we denote the (closed, symmetrical cartesian monoidal category) category of small categories with functors. The theory of (strict) 2-categories is (by definition) the theory of categories enriched over **Cat**. In this section, we spell out what this means. This section is based upon [2].
When working with (ordinary) categories, one usually restricts themselves to locally small categories, i.e. the hom-sets are sets. But sometimes the hom-sets have some additional structure. In the case of abelian categories, one has that the hom-sets are abelian groups and the composition is a group homomorphism. A 2-category is a category in which the hom-sets now have the structure of a category (and the composition is a functor instead of merely a function). So besides having only morphisms between the objects, we also have morphisms between the morphisms, these are called 2-cells.
The product of categories is denoted by $\times$ and the terminal category (i.e. the category with 1 object and only the trivial morphism) is denoted by $\mathbf{1_{Cat}}$.

Definition 1. *A 2-category $\mathcal{K}$ consists of:*

- *a class of objects (or 0-cells) $\mathcal{K}_0$,*
- *for each $A, B \in \mathcal{K}_0$, a small category $\mathcal{K}(A, B)$ whose objects (resp. morphisms) are called 1-cells or arrows (resp. 2-cells),*
- *for each $A, B, C \in \mathcal{K}_0$, a functor*

$$c = c_{A,B,C} : \mathcal{K}(A, B) \times \mathcal{K}(B, C) \to \mathcal{K}(A, C),$$

  called the composition,

- *for each $A \in \mathcal{K}_0$, a functor*

$$u = u_A : \mathbf{1_{Cat}} \to \mathcal{K}(A, A),$$

  called the unit.

These data must satisfy the following commutativity axioms:

$$
\begin{array}{ccccc}
\mathbf{1_{Cat}} \times \mathcal{K}(A,B) & \xrightarrow{\cong} & \mathcal{K}(A,B) & \xleftarrow[\cong]{} & \mathcal{K}(A,B) \times \mathbf{1_{Cat}} \\
\downarrow{\scriptstyle u_A \times Id} & & \downarrow{\scriptstyle Id} & & \downarrow{\scriptstyle Id \times u_B} \\
\mathcal{K}(A,A) \times \mathcal{K}(A,B) & \xrightarrow{c_{AAB}} & \mathcal{K}(A,B) & \xleftarrow[c_{ABB}]{} & \mathcal{K}(A,B) \times \mathcal{K}(B,B)
\end{array}
$$

$$
\begin{array}{ccc}
(\mathcal{K}(A,B) \times \mathcal{K}(B,C)) \times \mathcal{K}(C,D) & \xrightarrow{c_{ABC} \times Id} & \mathcal{K}(A,C) \times \mathcal{K}(C,D) \\
\downarrow{\scriptstyle \cong} & & \\
\mathcal{K}(A,B) \times (\mathcal{K}(B,C) \times \mathcal{K}(C,D)) & & \downarrow{\scriptstyle c_{ACD}} \\
\downarrow{\scriptstyle Id \times c_{BCD}} & & \\
\mathcal{K}(A,B) \times \mathcal{K}(B,D) & \xrightarrow{c_{ABD}} & \mathcal{K}(A,D)
\end{array}
$$

Note that the unit functor u_A corresponds with both a 1-cell which we denote by Id_A and a 2-cell which we denote by Id_{Id_A}. The first diagram means that Id_A (resp. Id_{Id_A}) acts as a unit under the composition of 1-cells (resp. 2-cells). And the commutativity of the second diagram means that the composition (for both 1-and 2-cells) is associative.

In a 2-category, a 1-cell is represented by an arrow $f : A \to B$, a 2-cell α from $f : A \to B$ to $g : A \to B$ will be denoted by $\alpha : f \Rightarrow g : A \to B$ and will be visualized as

$$A \underset{g}{\overset{f}{\Downarrow \alpha}} B \ .$$

Since each $\mathcal{K}(A,B)$ is a category, 2-cells can be composed *vertically*: If $\alpha : f \Rightarrow g : A \to B$ and $\beta : g \Rightarrow h$ are 2-cells, we can compose them to have $A \underset{h}{\overset{f}{\Downarrow \beta\circ\alpha}} B$. By functoriality of the composition, we can also compose *horizontally*:

$$A \underset{g}{\overset{f}{\Downarrow \alpha}} B \underset{\tilde{g}}{\overset{\tilde{f}}{\Downarrow \tilde{\alpha}}} C \quad = \quad A \underset{\tilde{g}\circ g}{\overset{\tilde{f}\circ f}{\Downarrow \tilde{\alpha}\bullet\alpha}} C \ .$$

That the composition is functorial means that it does not matter whether we first compose vertical and then horizontal or vice versa, i.e. consider the following 2-cells:

$$A \underset{g}{\overset{f}{\Downarrow \alpha}} B \ , \ A \underset{h}{\overset{g}{\Downarrow \beta}} B \ , \ B \underset{\tilde{g}}{\overset{\tilde{f}}{\Downarrow \tilde{\alpha}}} C \ , \ B \underset{\tilde{g}}{\overset{\tilde{f}}{\Downarrow \tilde{\alpha}}} C \ .$$

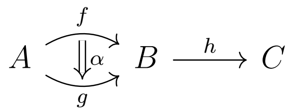
Then

$$(\tilde{\beta} \circ \tilde{\alpha}) \bullet (\beta \circ \alpha) = (\tilde{\beta} \bullet \beta) \circ (\tilde{\alpha} \bullet \alpha). \tag{1.1}$$

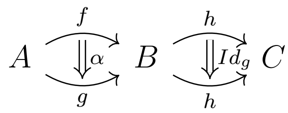
i.e. the following diagram is well-defined:

$$
\begin{array}{ccccc}
 & \overset{f}{\frown} & & \overset{\tilde{f}}{\frown} & \\
 & \Downarrow \alpha & & \Downarrow \tilde{\alpha} & \\
A & \xrightarrow{\ g\ } & B & \xrightarrow{\ \tilde{g}\ } & C \\
 & \Downarrow \beta & & \Downarrow \tilde{\beta} & \\
 & \underset{h}{\smile} & & \underset{\tilde{h}}{\smile} &
\end{array}
$$

This rule will be referred to as the *interchange law.*
Consider $\alpha : f \Rightarrow g : A \to B$ and $h : B \to C$ a 1-cell. The diagram



will be used as notation for



We clearly have that each locally small category has the structure of a 2-category with only trivial 2-cells.

Just as **Set** is the archetypal 1-category, the category of all small categories is the archetypal 2-category:

Example 1. *The category* **Cat** *has the structure of a* 2*-category where the* 2*-cells are the natural transformations.*

Proof. We already now that **Cat** is a (1−)category and that $\mathbf{Cat}(A, B)$ forms a small category whose objects are functors and whose morphisms are the natural transformations (for all small categories A, B) where the composition is given by the component wise composition. We now define the horizontal composition and unit functor:

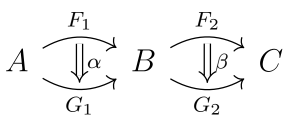
- We have to define a functor

  $$\bullet : \mathbf{Cat}(A, B) \times \mathbf{Cat}(B, C) \to \mathbf{Cat}(A, C).$$

  At the level of the objects, we define this functor as applying the composition, i.e.

  $$(A \xrightarrow{F} B, B \xrightarrow{G} C) \mapsto G \bullet F := G \circ F.$$

  We now define $\bullet$ at the level of the morphisms, so consider the following natural transformations:

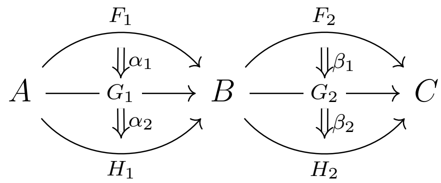
  The composition $\beta \bullet \alpha : F_2 \circ F_1 \to G_2 \circ G_1$ is given (componentwise) by:

  $$F_2(F_1(a)) \xrightarrow{F_2(\alpha_a)} F_2(G_1(a)) \xrightarrow{\beta_{G_1(a)}} G_2(G_1(a)), \quad \forall a \in A.$$

  That $\beta \bullet \alpha$ is indeed a natural transformation follows because the following diagram commutes:

  $$\begin{array}{ccccc} F_2(F_1(a)) & \xrightarrow{F_2(\alpha_a)} & F_2(G_1(a)) & \xrightarrow{\beta_{G_1(a)}} & G_2(G_1(a)) \\ {\scriptstyle F_2(F_1(f))}\downarrow & & {\scriptstyle F_2(G_1(f))}\downarrow & & \downarrow{\scriptstyle G_2(G_1(f))} \\ F_2(F_1(b)) & \xrightarrow[F_2(\alpha_b)]{} & F_2(G_1(b)) & \xrightarrow[\beta_{G_1(b)}]{} & G_2(G_1(b)) \end{array}$$

Indeed: The left diagram commutes because by naturality of α we have $G_1(f)\circ \alpha_a = \alpha_b \circ F_1(f)$ and then applying the functoriality of F_2. The right diagram commutes by naturality of β (since $G_1(f)$ is a morphism $G_1(a) \to G_1(b)$).
We now show that this composition is functorial, so we first show that the following diagram is well-defined:



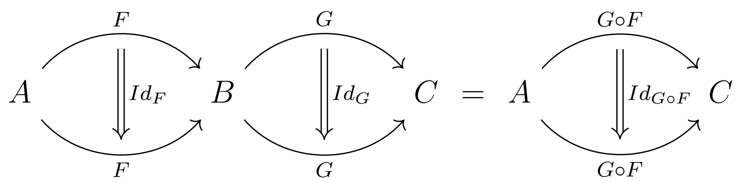
So for each $a \in A$, we have to show:

$$\begin{array}{ccc} F_2(F_1(a)) & & F_2(F_1(a)) \\ \downarrow{\scriptstyle F_2((\alpha_1)_a)} & & \downarrow{\scriptstyle F_2((\alpha_1)_a)} \\ F_2(G_1(a)) & & F_2(G_1(a)) \\ \downarrow{\scriptstyle (\beta_1)_{G_1(a)}} & & \downarrow{\scriptstyle F_2((\alpha_2)_a)} \\ G_2(G_1(a)) & = & F_2(H_1(a)) \\ \downarrow{\scriptstyle G_2((\alpha_2)_a)} & & \downarrow{\scriptstyle (\beta_1)_{H_1(a)}} \\ G_2(H_1(a)) & & G_2(H_1(a)) \\ \downarrow{\scriptstyle (\beta_2)_{H_1(a)}} & & \downarrow{\scriptstyle (\beta_2)_{H_1(a)}} \\ H_2(H_1(a)) & & H_2(H_1(a)) \end{array}$$

Since the first and last morphisms are equal, it reduces to show

$$G_2((\alpha_2)_a) \circ (\beta_1)_{G_1(a)} = (\beta_1)_{H_1(a)} \circ F_2((\alpha_2)_a).$$

This equality indeed holds by naturality of α_2.
To conclude the functoriality, we have to show that the unit 2-cell is preserved, i.e.



This clearly holds because the left natural transformation is component-wise given by

$$G(F(a)) \xrightarrow{G((Id_F)_a)} G(F(a)) \xrightarrow{(Id_G)_{F(a)}} G(F(a))$$

and both equal $Id_{GF(a)} : GF(a) \to GF(a)$.
So all together we conclude that

$$\bullet : \mathbf{Cat}(A, B) \times \mathbf{Cat}(B, C) \to \mathbf{Cat}(A, C),$$

is indeed a functor.

- The unit functor

$$u_A : \mathbf{1_{Cat}} \to \mathbf{Cat}(A, A)$$

is defined as

$$\star \mapsto Id_A, \quad Id_\star \mapsto Id_{Id_A}$$

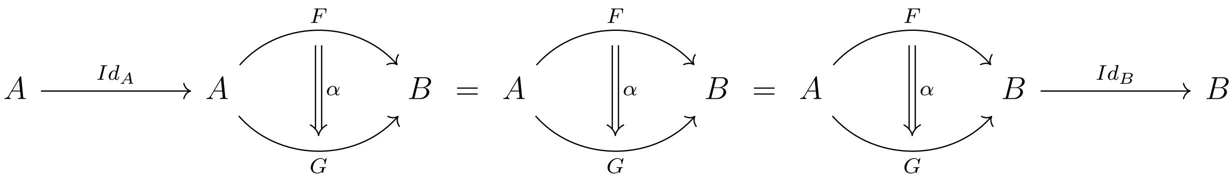
where Id_{Id_A} is the natural transformation defined as $(Id_{Id_A})_a := Id_a$ (for each $a \in A$). This clearly is a 2-functor.

We now show the commutativity conditions which shows that **Cat** is indeed a 2-category.

- **Unit condition:** We have to show that the following diagram commutes:

$$\begin{array}{ccccc}
\mathbf{1_{Cat}} \times \mathbf{Cat}(A,B) & \xrightarrow{\cong} & \mathbf{Cat}(A,B) & \xleftarrow[\cong]{} & \mathbf{Cat}(A,B) \times \mathbf{1_{Cat}} \\
\downarrow{\scriptstyle u_A \times Id} & & \downarrow{\scriptstyle Id} & & \downarrow{\scriptstyle Id \times u_B} \\
\mathbf{Cat}(A,A) \times \mathbf{Cat}(A,B) & \xrightarrow{\bullet} & \mathbf{Cat}(A,B) & \xleftarrow{\bullet} & \mathbf{Cat}(A,B) \times \mathbf{Cat}(B,B)
\end{array}$$

For an object $a \in A$, we have $u_A(a) = Id_a$ (the identity morphism on a), thus this commutativity is clearly satisfied at the level on the objects. To show it at the level of the morphisms, we have to show:



i.e. we have to show for each $a \in A$

$$\alpha_{Id_A(a)} \circ F(Id_A(a)) = \alpha_{Id_A(a)} = (Id_A)_{G(a)} \circ Id_B(\alpha_a),$$

but this is clear.

- **Associativity condition**: We have to show that the following diagram commutes:

$$\begin{array}{ccc}
(\mathbf{Cat}(A,B) \times \mathbf{Cat}(B,C)) \times \mathbf{Cat}(C,D) & \xrightarrow{\bullet} & \mathbf{Cat}(A,C) \times \mathbf{Cat}(C,D) \\
\downarrow{\scriptstyle \cong} & & \\
\mathbf{Cat}(A,B) \times (\mathbf{Cat}(B,C) \times \mathbf{Cat}(C,D)) & & \downarrow{\scriptstyle \bullet} \\
\downarrow{\scriptstyle Id \times \bullet} & & \\
\mathbf{Cat}(A,B) \times \mathbf{Cat}(B,D) & \xrightarrow{\bullet} & \mathbf{Cat}(A,D)
\end{array}$$

Since the composition of functors is associative, we have this commutativity on the level of the objects (i.e. functors). So it remains to check the commutativity on the level of the morphisms (i.e. the natural transformations). Consider the following natural transformations:

$$A \underset{G_1}{\overset{F_1}{\Downarrow \alpha}} B \underset{G_2}{\overset{F_2}{\Downarrow \beta}} C \underset{G_3}{\overset{F_3}{\Downarrow \gamma}} D$$

We first spell out $\gamma \bullet (\beta \bullet \alpha)$ and $(\gamma \bullet \beta) \bullet \alpha$. Let $a \in A$, then:

$$\begin{aligned}
(\gamma \bullet (\beta \bullet \alpha))_a &= F_3(F_2F_1a) \xrightarrow{F_3(\alpha_a)} F_3(G_2G_1a) \xrightarrow{\gamma_{G_2G_1a}} F_3(F_2F_1a) \\
&= F_3(F_2F_1a) \xrightarrow{F_3F_2(\alpha_a)} F_3(F_2G_1a) \xrightarrow{F_3(\beta_{G_1a})} F_3(G_2G_1a) \xrightarrow{\gamma_{G_2G_1a}} G_3(G_2G_1a) \\
((\gamma \bullet \beta) \bullet \alpha)_a &= F_3(F_2F_1a) \xrightarrow{F_3F_2(\alpha_a)} F_3F_2(G_1a) \xrightarrow{(\gamma\bullet\beta)_{G_2G_1a}} G_3(G_2G_1a) \\
&= FF_3(F_2F_1a) \xrightarrow{F_3F_2(\alpha_a)} F_3(F_2G_1a) \xrightarrow{F_3(\beta_{G_1a})} F_3(G_2G_1a) \xrightarrow{\gamma_{G_2G_1a}} G_3(G_2G_1a)
\end{aligned}$$

So they are indeed the same, which shows the associativity of $\bullet$. □

Example 2. *If $\mathcal{K}$ is a 2-category, its opposite category $\mathcal{K}^{op}$ has the same objects and $\mathcal{K}^{op}(A,B) := \mathcal{K}(B,A)$ where the composition functors*

$$c^{op}_{ABC} : \mathcal{K}^{op}(A,B) \times \mathcal{K}^{op}(B,C) \to \mathcal{K}^{op}(A,C)$$

are given by

$$s^{CB}_{BA} : \mathcal{K}(B,A) \times \mathcal{K}(C,B) \overset{\cong}{\to} \mathcal{K}(C,B) \times \mathcal{K}(B,A) \xrightarrow{c_{C,B,A}} \mathcal{K}(C,A).$$

So we have that $\mathcal{K}^{op}$, has the same 0-and 2-cells, but the 1-cells are reversed.

Definition 2. *Let $\mathcal{K}$ and $\mathcal{L}$ be 2-categories. A 2-**functor** F from $\mathcal{K}$ to $\mathcal{L}$, denoted by $F : \mathcal{K} \to \mathcal{L}$, assigns to each 0-cell A in $\mathcal{K}$, a 0-cell FA in $\mathcal{L}$ and for 0-cells A and B in $\mathcal{K}$, a functor*

$$F = F_{AB} : \mathcal{K}(A,B) \to \mathcal{L}(FA,FB)$$

such that its respects composition and the unit, more precisely, the following diagrams commute:

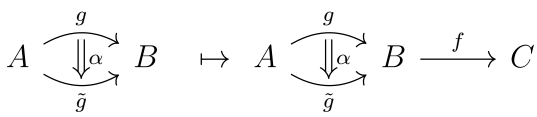
$$\begin{array}{ccc}
\mathcal{K}(A,B) \times \mathcal{K}(B,C) & \xrightarrow{c_{ABC}} & \mathcal{K}(A,C) \\
\downarrow{\scriptstyle F_{AB} \times F_{BC}} & & \downarrow{\scriptstyle F_{AC}} \\
\mathcal{L}(FA,FB) \times \mathcal{L}(FB,FC) & \xrightarrow[c_{FA,FB,FC}]{} & \mathcal{L}(FA,FC)
\end{array}
\qquad
\begin{array}{ccc}
\mathbf{1_{Cat}} & \xrightarrow{u_A} & \mathcal{K}(A,A) \\
& \searrow{\scriptstyle u_{FA}} & \downarrow{\scriptstyle F_{AA}} \\
& & \mathcal{L}(FA,FA)
\end{array}$$

So a 2-functor $F : \mathcal{K} \to \mathcal{L}$ assigns to each 0-cell A in $\mathcal{K}$, a 0-cell FA in $\mathcal{L}$, a 1-cell $f : A \to B$ in $\mathcal{K}$ to a 1-cell $F_{AB}(f) = F(f) : FA \to FB$ in $\mathcal{L}$ and a 2-cell $\alpha : f \Rightarrow g : A \to B$ in $\mathcal{K}$, a 2-cell $F(\alpha) = F_{AB}(\alpha) : Ff \Rightarrow Fg : FA \to FB$ and the functoriality of F_{AB} means that F of the identity 2-cell is the identity 2-cell (between the image of the 1-cell and itself) and F preserves the vertical composition.

Example 3. *(”**Representable functors**”) Let $\mathcal{K}$ be a 2-category and $A \in \mathcal{K}$ a 0-cell. The hom-categories of A induces 2-functors:*
*The covariant one $\mathcal{K}(A,-) : \mathcal{K} \to$ **Cat** maps a 0-cell B (in $\mathcal{K}$) to the small category*

$\mathcal{K}(A,B)$. *A* 1*-cell* $f : B \to C$ *is mapped to the* 1*-cell in* **Cat** *(i.e. a functor) defined by:*

$$\begin{aligned}(f \circ -) \quad &: \mathcal{K}(A,B) &&\to \mathcal{K}(A,C)\\ &\quad g &&\mapsto f \circ g\end{aligned}$$

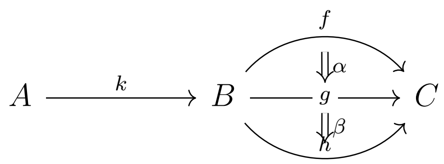


A 2*-cell* $\alpha : f \Rightarrow g : B \to C$ *is mapped to the natural transformation given by* $\left(f \circ h \xrightarrow{\alpha \circ h} g \circ h\right)_{h \in \mathcal{K}(A,B)}$.

Proof. We clearly have that $f \circ -$ is a functor and $\alpha \circ h$ is a natural transformation. That

$$\mathcal{K}(A,-)_{BC} : \mathcal{K}(B,C) \to \mathbf{Cat}\left(\mathcal{K}(A,B), \mathcal{K}(A,C)\right),$$

is a functor is immediate because if we have the following data:



Then we have

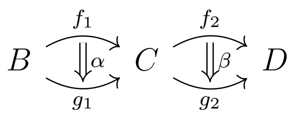
$$\begin{aligned}\left(\mathcal{K}(A,\beta) \circ \mathcal{K}(A,\alpha)\right)_k &= \mathcal{K}(A,\beta)_k \circ \mathcal{K}(A,\alpha)_k\\ &= (\beta \bullet Id_k) \circ (\alpha \bullet Id_k)\\ &= (\beta \circ \alpha) \bullet (Id_k \circ Id_k) = (\beta \circ \alpha) \bullet Id_k\\ &= \mathcal{K}(A, \beta \circ \alpha)_k\end{aligned}$$

where the third equality holds by the interchange law in $\mathcal{K}$. So we indeed have that $\mathcal{K}(A,-)_{BC}$ preserves the composition, that it preserves the identity follows in an analoguous way using $Id_f \bullet Id_k = Id_{f \circ k}$.

So now it remains to check the associativity and unital condition of $\mathcal{K}(A,-)$. The associativity condition means that the following diagram commutes:

$$\begin{array}{ccc}\mathcal{K}(B,C) \times \mathcal{K}(C,D) & \xrightarrow{\quad c_{BCD} \quad} & \mathcal{K}(B,D)\\ \Big\downarrow{\scriptstyle \mathcal{K}(A,-)_{BC} \times \mathcal{K}(A,-)_{CD}} & & \Big\downarrow{\scriptstyle \mathcal{K}(A,-)_{BD}}\\ \mathbf{Cat}(\mathcal{K}(A,B),\mathcal{K}(A,C)) \times \mathbf{Cat}(\mathcal{K}(A,C),\mathcal{K}(A,D)) & \xrightarrow[c]{} & \mathbf{Cat}(\mathcal{K}(A,B),\mathcal{K}(A,D))\end{array}$$

This commutativity clearly holds, indeed: At the level of the objects (i.e. 1-cells in $\mathcal{K}$), we have that the image of $(B \xrightarrow{f} C, C \xrightarrow{g} D)$ is given by the functor $\mathcal{K}(A, g \circ f) = (g \circ f) \circ -$. At the level of objects (i.e. 2-cells in $\mathcal{K}$), we have that the image of



is given by the natural transformation

$$\left(f_2 \circ f_1 \circ h \xrightarrow{(\beta \bullet \alpha) \bullet h} g_2 \circ g_1 \circ h\right)_{h \in \mathcal{K}(A,B)}.$$

To show the unit axiom, we have to show that the following diagram commutes:

$$\begin{array}{ccc} \mathbf{1_{Cat}} & \xrightarrow{u_B} & \mathcal{K}(B,B) \\ & \searrow^{u_{\mathcal{K}(A,B)}} & \downarrow{\scriptstyle \mathcal{K}(A,-)_{BB}} \\ & & \mathbf{Cat}(\mathcal{K}(A,B), \mathcal{K}(A,B)) \end{array}$$

This also clearly holds because u_B maps the (unique) object to the identity 1-cell Id_B and the (unique) morphism to the identity 2-cell Id_{Id_B} and $u_{\mathcal{K}(A,B)}$ maps the (unique) object to the identity functor $Id_{\mathcal{K}(A,B)}$ and the (unique) morphism to the identity natural transformation which is objectwise given by the identity. □

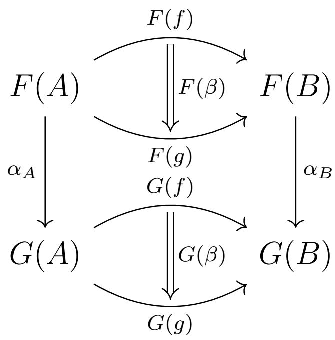
Analogously, we also have the contravariant representable:

$$\mathcal{K}(-, A) : \mathcal{K}^{op} \to \mathbf{Cat}.$$

Definition 3. *Let $\mathcal{K}$ and $\mathcal{L}$ be 2-categories, $F, G : \mathcal{K} \to \mathcal{L}$ be 2-functors. A **2-natural transformation** α, denoted by $\alpha : F \implies G$, assigns to each 0-cell A in $\mathcal{K}$ a functor $\alpha_A : \mathbf{1_{Cat}} \to \mathcal{L}(FA, GA)$ such that for all 0-cells $A, B \in \mathcal{K}$, the following diagram commutes:*

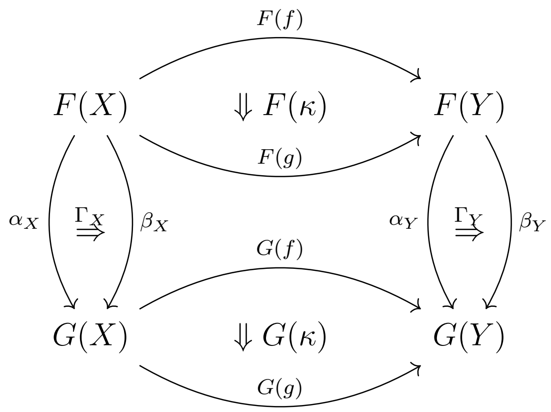
$$\begin{array}{ccc} \mathbf{1_{Cat}} \times \mathcal{K}(A,B) & \xrightarrow{\cong} & \mathcal{K}(A,B) \times \mathbf{1_{Cat}} \\ \downarrow{\scriptstyle \alpha_A \times G_{AB}} & & \downarrow{\scriptstyle F_{AB} \times \alpha_B} \\ \mathcal{L}(FA,GA) \times \mathcal{L}(GA,GB) & & \mathcal{L}(FA,FB) \times \mathcal{L}(FB,GB) \\ & \searrow^{c_{FA,GA,GB}} \quad \swarrow_{c_{FA,FB,GB}} & \\ & \mathcal{L}(FA,GB) & \end{array}$$

Unwrapping this definition, this means that α consists of 1-cells $\alpha_A : FA \to GA$ in $\mathcal{L}$ such that for every 2-cell $\beta : f \Rightarrow g : A \to B$, we have that the following diagram commutes:



So in particular we also need $G(f) \circ \alpha_A = \alpha_B \circ F(f)$.

Definition 4. *Let $\alpha, \beta : F \implies G$ be 2-natural transformations. A modification Γ from α to β consists of 2-cells $\Gamma_X : \alpha_X \to \beta_X$ (for each 0-cell $X \in \mathcal{A}$) such that if $\kappa : f \implies g : X \to Y$ is a 2-cell in $\mathcal{A}$, the following diagram commutes:*



Example 4. *Let $F, G : \mathcal{A} \to \mathcal{B}$ be 2-functors. The 2-natural transformations from F to G and modifications between them form a category where the composition of modifications is given by the object-wise composition of 2-cells, more precisely: If $\Gamma : \alpha \to \beta$ and $\tilde{\Gamma} : \beta \to \gamma$ are modifications (with α, β and γ natural transformations $F \to G$). The composition $\tilde{\Gamma} \circ \Gamma$ is defined by $\left(\tilde{\Gamma} \circ \Gamma\right)_X := \tilde{\Gamma}_X \circ \Gamma_X$ (notice that this is the vertical composition in $\mathcal{B}$). The identity modification just consists of the identity 2-cells.*

Proof. Clearly the identity modification defined as consisting of the identity 2-cells is a modification. That the composition $\tilde{\Gamma} \circ \Gamma$ is indeed a modification follows because $G(\kappa) \circ \tilde{\Gamma}_X \circ \Gamma_X = \tilde{\Gamma}_Y \circ \Gamma_Y \circ F(\kappa)$. Indeed: Since Γ (resp. $\tilde{\Gamma}$) is a modification, we have:

$$G(\kappa) \circ \Gamma_X = \Gamma_Y \circ F(\kappa), \quad G(\kappa) \circ \tilde{\Gamma}_X = \tilde{\Gamma}_Y \circ F(\kappa).$$

The result now follows because of the interchange law (in $\mathcal{B}$). □

Example 5. *The category $Fun_2(\mathcal{A}, \mathcal{B})$, of 2-functors and 2-natural transformations has the structure of a 2-category whose 2-cells are the modifications:*

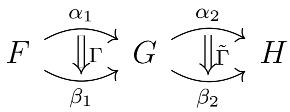
Proof. This proof is actually the same as showing that the (small) categories, together with the functors and natural transformations form a 2-category because a modification between natural transformation is defined analogous as a natural transformation between functors is defined. Therefore we just give the construction:
Let $F, G : \mathcal{A} \to \mathcal{B}$ be 2-functors. By the previous example we know that the 2-natural transformations together with the modifications indeed forms a category. So we have to define the composition functors

$$c : Fun_2(\mathcal{A}, \mathcal{B})(F, G) \times Fun_2(\mathcal{A}, \mathcal{B})(G, H) \to Fun_2(\mathcal{A}, \mathcal{B})(F, H),$$

and unit functors:

$$u : \mathbf{1_{Cat}} \to Fun_2(\mathcal{A}, \mathcal{B})(F, F).$$

The unit functor u maps the (unique) object to the identity 2-natural transformation (i.e. consists of the identity 1-cells) and the (unique) morphism is mapped to the modification which consists of the identity 2-cells.
The composition functor maps natural transformations ($\alpha : F \to G, \beta : G \to H$) to the vertical composition of natural transformations (i.e. $(\beta \circ \alpha)_X := \beta_X \circ \alpha_X$). The composition of modifications



is given by the modification which is object-wise given as:

$$\beta_X \circ \alpha_X \xrightarrow{Id_{\beta_X} \circ \Gamma_X} \beta_X \circ \tilde{\alpha}_X \xrightarrow{\tilde{\Gamma}_X \circ Id_{\tilde{\alpha}_X}} \tilde{\beta}_X \circ \tilde{\alpha}_X, \quad \forall X \in \mathcal{A}.$$

□

1.2 Yoneda lemma

The (probably) most important theorem in (ordinary) category theory is the *Yoneda lemma* which says that for a functor $F : \mathcal{C} \to \mathbf{Set}$ (with $\mathcal{C}$ locally small), there is a bijection

$$Nat(\mathcal{C}(X, -), F) \cong F(X),$$

which is moreover natural in both X and F. In this section we show that this bijection extends to an isomorphism of categories when working in the 2-categorical setting.

Let $F, G : \mathcal{A} \to \mathcal{B}$ be 2-functors. The 2-natural transformations and 2-modifications form a $(1-)$category which we denote by $Nat_2(F, G)$. The goal of this section is to show:

Theorem 1. *(”**Yoneda Lemma**”) Let $F : \mathcal{A} \to \mathbf{Cat}$ be a 2-functor. Then there is an isomorphism (of 1-categories)*

$$Nat_2(\mathcal{A}(A, -), F) \cong F(A),$$

which is natural in A and F.

In this section we fix some 2-functor $F : \mathcal{A} \to \mathbf{Cat}$ and a 0-cell $A \in \mathcal{A}$.

Lemma 1. *The assignments*

$$\begin{aligned} \alpha : \mathcal{A}(A, -) \to F &\quad\mapsto\quad \alpha_A(Id_A) \\ \Gamma : \alpha \to \beta : \mathcal{A}(A, -) \to F &\quad\mapsto\quad (\Gamma_A)_{Id_A} \end{aligned}$$

define a functor

$$\phi : Nat_2(\mathcal{A}(A, -), F) \to F(A).$$

Proof. First notice that this is well-defined because:

- $FA \in \mathbf{Cat}$, hence $\alpha_A(Id_A)$ is an element in $F(A)$.
- $\Gamma : \alpha \to \beta$ is a modification, thus $\Gamma_A : \alpha_A \to \beta_A$ is a 2-natural transformation. Thus $(\Gamma_A)_{Id_A} : \alpha_A(Id_A) \to \beta_A(Id_A)$ is a morphism in $F(A)$.

That ϕ is functorial is immediate by the following computation:

$$\begin{aligned} \phi(\Delta \circ \Gamma) = ((\Delta \circ \Gamma)_A)_{Id_A} = (\Delta_A \circ \Gamma_A)_{Id_A} = (\Delta_A)_{Id_A} \circ (\Gamma)_{Id_A} = \phi(\Delta) \circ \phi(\Gamma) \\ \phi(Id_\Gamma) = ((Id)_A)_{Id_A} = Id_{\phi(\Gamma)} \end{aligned}$$

□

Lemma 2. *The following assignments define a functor* $\psi : FA \to Nat_2(\mathcal{A}(A,-),F)$*:*

- *An object* $a \in FA$ *is mapped to the* 2*-natural transformation* $\alpha^a : \mathcal{A}(A,-) \Longrightarrow F$*, which is object-wise given by:*

$$\psi(a)_X := F(-)(a) =: \alpha^a_X : \mathcal{A}(A,X) \to FX : z \mapsto F(z)(a), (g : z \to w) \mapsto F(g)_a,$$

for each $X \in \mathcal{A}$.

- *A morphism* $f \in F(A)(a,b)$ *(with* $a, b \in FA$*) is mapped to the modification* $\Gamma^f : \alpha^a \Longrightarrow \alpha^b$*, which is given by the* 2*-natural transformation* $\Gamma^f_X : \alpha^a_X \Longrightarrow \alpha^b_X$ *which is defined as*

$$\Big(F(z)(f) := \Gamma^f_X(z) : F(z)(a) = \alpha^a_X(z) \to F(z)(b) = \alpha^b_X(z)\Big)_{z \in \mathcal{A}(A,X)}.$$

Proof. We first show that this is well-defined.

- α^a_X defines a 2-natural transformation because for 1-cells $f, g \in \mathcal{A}(X,Y)$ and 2-cell $\beta \in \mathcal{A}(A,X)(f,g)$, the following diagrams commute:

$$\begin{array}{ccc} \mathcal{A}(A,X) & \xrightarrow{\alpha^a_X} & FX \\ {\scriptstyle f\circ-}\Big\downarrow \overset{\beta\circ-}{\Longrightarrow} \Big\downarrow{\scriptstyle g\circ-} & & {\scriptstyle Ff}\Big\downarrow \overset{F\beta}{\Longrightarrow} \Big\downarrow{\scriptstyle Fg} \\ \mathcal{A}(A,Y) & \xrightarrow{\alpha^a_Y} & FY \end{array} \qquad \begin{array}{ccc} \mathcal{A}(A,X) & \xrightarrow{\alpha^a_X} & FX \\ \Big\downarrow{\scriptstyle f\circ-} & & \Big\downarrow{\scriptstyle F(f)} \\ \mathcal{A}(A,Y) & \xrightarrow{\alpha^a_Y} & FY \end{array}$$

  Indeed:

$$\alpha^a_Y(f \circ z) = F(f \circ z)(a) = F(f)(F(z)(a)) = F(f)(\alpha^a_X(z)),$$
$$\alpha^a_Y(\beta \circ z) = F(\beta \circ Id_z)(a) = F(\beta)(F(Id_z)(a)) = F(\beta)(\alpha^a_X(z)).$$

- Γ^f_X defines a modification because for each 2-cell $\beta : f \Rightarrow g : X \to Y$ in $\mathcal{A}$, we have:

$$F(\beta) \circ \Gamma^f_X = \Gamma^f_Y \circ \mathcal{A}(A,\beta).$$

  Indeed: For each 1-cell $z \in \mathcal{A}(A,X)$ we have

$$F(\beta) \circ \Gamma^f_X(z) = F(\beta) \circ F(z)(f) = F(\beta \circ z)(f) = \Gamma^f_Y(\beta \circ z).$$

We now show that ψ is functorial. We first show that it preserves the identity, i.e. we have to show that for each $a \in F(A)$, we need that $\psi(Id_a)$ is the identity modification, i.e. we need to show that

$$\forall X \in \mathcal{A}, \forall z \in \mathcal{A}(A,X) : \Gamma^{Id_a}_X(z) = Id_{F(z)(a)}.$$

That this holds follows because:

$$\Gamma^{Id_a}_X(z) = F(z)(Id_a) = Id_{F(z)(a)}.$$

The first equality holds by definition of $\Gamma_X^{Id_a}$ and the second holds because $F(z) : F(A) \to F(X)$ is a functor.
To show that it preserves the composition, let $f \in F(A)(a,b)$ and $g \in F(A)(b,c)$. To show $\psi(g \circ f) = \psi(g) \circ \psi(f)$, we have to show that

$$\forall X \in \mathcal{A}, \forall z \in \mathcal{A}(A,X) : \Gamma_X^{g \circ f}(z) = \Gamma_X^g(z) \circ \Gamma_X^f(z).$$

That this holds follows because

$$\Gamma_X^{g \circ f}(z) = F(z)(g \circ f) = F(z)(g) \circ F(z)(f) = \Gamma_X^g(z) \circ \Gamma_X^f(z).$$

□

Lemma 3. *ϕ and ψ are inverses to eachother.*

Proof. First notice that we have $\phi \circ \psi = Id_{F(A)}$ because for each $a, b \in F(A)$ and $f \in F(A)(a,b)$ we have:

$$\begin{aligned}(\phi \circ \psi)(a) &= \phi(\alpha^a) = (\alpha_A^a)(Id_A) = F(Id_A)(a) = a \\ (\phi \circ \psi)(f) &= \phi(\Gamma^f) = (\Gamma_A^f)_{Id_A} = F(Id_A)(f) = f\end{aligned}$$

We now have to show $\psi \circ \phi = Id_{Nat_2(\mathcal{A}(A,-),F)}$.
So first consider a 2-natural transformation $\alpha : \mathcal{A}(A,-) \to F$. We have to show $\psi(\phi(\alpha))_B = \alpha_B$ for all 0-cells $B \in \mathcal{A}$. So we have to show

$$\forall f \in \mathcal{A}(A,B) : \psi(\phi(\alpha))_B(f) = \alpha_B(f).$$

This follows by the following computation:

$$\begin{aligned}\psi(\phi(\alpha))_B(f) &= \psi(\alpha_A(Id_A))_B(f), \quad \text{by definition } \phi \\ &= F(f)(\alpha_A(Id_A)), \quad \text{by definition } \psi \\ &= \alpha_B(f \circ Id_A), \quad \text{by naturality } \alpha \\ &= \alpha_B(f)\end{aligned} \tag{1.2}$$

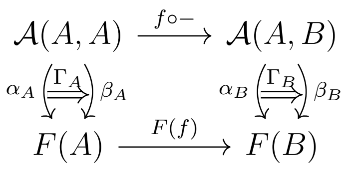
Now consider a modification $\Gamma : \alpha \to \beta$. So we now have to show

$$\forall f \in \mathcal{A}(A,B) : \psi(\phi(\Gamma))_B(f) = \Gamma_B(f).$$

This follows by the same computation:

$$\begin{aligned}\psi(\phi(\Gamma))_B(f) &= F(f)((\Gamma_A)_{Id_A}) \\ &= \Gamma_B(f \circ Id_A) = \Gamma_B(f)\end{aligned}$$

where the second equality holds since Γ is a modification, indeed: Since Γ is a modification, we have that the following diagram commutes:



So applying this to Id_A, we get the desired equality. $\square$

We now conclude the Yoneda lemma by showing the naturality:

Lemma 4. *The isomorphism (of* 1*-categories)*

$$Nat_2(\mathcal{A}(A,-),F) \cong F(A),$$

is natural in A and F.

Proof. We have to show that the following diagram commutes (for each 1-cell $f \in \mathcal{A}(A,B)$):

$$\begin{array}{ccc} Nat_2(\mathcal{A}(A,-),F) & \xrightarrow{\cong} & FA \\ {\scriptstyle Nat_2(\mathcal{A}(f,-),F)}\big\downarrow & & \big\downarrow{\scriptstyle Ff} \\ Nat_2(\mathcal{A}(B,-),F) & \xrightarrow{\cong} & FB \end{array}$$

Let $\alpha : \mathcal{A}(A,-) \to F$ be a 2-natural transformation. We have (by definition) $(Nat_2(\mathcal{A}(f,-),F)(\alpha))_X := \alpha_X(- \circ f)$. So the lower path is (by definition of the isomorphism) given by $\alpha_B(f)$ and we showed in (1.2) that this equals $F(f)(\alpha_A(Id_A))$ which is precisely the upper path. That the naturality also holds for modifications is exactly the same. $\square$

Definition 5. *Let $\mathcal{A}$ be a small* 2*-category. The Yoneda embedding is the* 2*-functor:*

$$\mathcal{Y} : \mathcal{A}^{op} \to Fun_2(\mathcal{A}, \mathbf{Cat}) : A \mapsto \mathcal{A}(A,-).$$

1.3 Weighted (co)limits

In this section we introduce the notion of weighted (co)limits. In (ordinary) category, the notion of (co)limits is usually expressed in terms of (co)cones. A more formal way of describing (co)limits is the following: Let $\mathcal{A}$ be a small category and let $F : \mathcal{A} \to \mathcal{B}$ be a functor. A cone for F is an object $B_0 \in \mathcal{B}$ together with a natural transformation

$$\Delta_{B_0} \Rightarrow F,$$

with

$$\Delta_{B_0} : \mathcal{A} \to \mathcal{B} : A \mapsto B_0.$$

Equivalently, a cone is given by a natural transformation

$$\Delta \Rightarrow \mathcal{B}\left(B_0, F(-)\right),$$

where

$$\Delta : \mathcal{A} \to \mathbf{Set} : A \mapsto \{\star\}.$$

The limit is then such an object and a natural transformation which is universal in the following sense:

Proposition 1. *The limit of F, if it exists, is an object* $\lim F$ *together with isomorphisms*

$$\mathcal{B}(B, \lim F) \cong \mathbf{Nat}(\Delta, \mathcal{B}(B, F-)),$$

which is natural in $B \in \mathcal{B}$.

Proof. We first show that the cones are indeed given by such natural transformations

$$\Delta \Rightarrow \mathcal{B}\left(B, F(-)\right).$$

Since $W(A)$ is the terminal category (for each 0-cell $A \in \mathcal{A}$), a natural transformation $\alpha : \Delta \to \mathcal{B}(B, F-)$ is given by a collection of morphisms $(\alpha_A : B \to FA)_{A \in \mathcal{A}}$. Moreover, this forms a cone for F i.e. for each morphism $f : A_1 \to A_2$, we have $F(f) \circ \alpha_{A_1} = \alpha_{A_2}$, indeed: By naturality of α, the following diagram commutes and this means precisely the equality of the equation:

$$\begin{array}{ccc} \Delta A_1 & \xrightarrow{\alpha_{A_1}} & \mathcal{B}(B, FA_1) \\ \downarrow{\scriptstyle \Delta f} & & \downarrow{\scriptstyle Ff\circ -} \\ \Delta A_2 & \xrightarrow{\alpha_{A_2}} & \mathcal{B}(B, FA_2) \end{array}$$

Conversely, if $(B, f) \equiv (B \xrightarrow{f_A} FA)_{A \in \mathcal{A}}$ is a cone for F, then this defines a natural transformation $\alpha^{(B,f)}$ with $\alpha_A^{(B,f)} := f_A$ (for $A \in \mathcal{A}$). The naturality of $\alpha^{(B,f)}$ means precisely the cone property, i.e. $F(g) \circ f_{A_1} = f_{A_2}$ (with $g : A_1 \to A_2 \in \mathcal{A}$).
We now show that if there exists an object $\lim F$ together with isomorphisms

$$\mathcal{B}(B, \lim F) \cong \mathbf{Nat}(\Delta, \mathcal{B}(B, F-)),$$

which is natural in $B \in \mathcal{B}$, the natural transformation α corresponding with $Id_{\lim F}$ under the natural isomorphism:

$$\mathbf{Nat}_2(\Delta, \mathcal{B}(B, F-)) \xrightarrow{\lambda_B} \mathcal{B}(B, \lim F), \quad B \in \mathcal{B},$$

is the universal cone for F. Consider a cone (B, f) for F and let $\alpha^{(B,f)}$ be the corresponding natural transformation. So we have a morphism

$$\lambda_B(\alpha^{(B,f)}) : B \to \lim F.$$

We claim that this is the unique morphism such that for each $A_1 \in \mathcal{A}$, we have $f_{A_1} = \alpha_{A_1} \circ \lambda_B(\alpha^{(B,f)})$. Since this should hold for each A_1, we have to show that the induced (2-)natural transformations are equal and this is indeed the case: By the universal property of the weighted limit, we have that the following diagram commutes:

$$\begin{array}{ccc} \mathbf{Nat}_2(W, \mathcal{B}(\lim F, F-)) & \xrightarrow{\lambda_{\lim F}} & \mathcal{B}(\lim F, \lim F) \\ {\scriptstyle \mathbf{Nat}_2(W, \mathcal{B}(\lambda_B(\alpha^{(B,f)}), F-))}\downarrow & & \downarrow{\scriptstyle -\circ\lambda_B(\alpha^{(B,f)})} \\ \mathbf{Nat}_2(W, \mathcal{B}(B, F-)) & \xrightarrow{\lambda_B} & \mathcal{B}(B, \lim F) \end{array}$$

The image of α along the path above is $\lambda_B(\alpha^{(B,f)})$ (since $\lambda_{\lim F}(\alpha) = Id_{\lim F}$) and the image of the path below is given by the image of

$$(B \xrightarrow{\lambda_B(\alpha^{(B,f)})} \lim F \xrightarrow{\alpha_A} FA)_{A \in \mathcal{A}}$$

under λ_B.
That $\lambda_B(\alpha^{(B,f)})$ is the unique morphism such that $f_{A_1} = \alpha_{A_1} \circ \lambda_B(\alpha^{(B,f)})$ holds follows by construction, indeed: By universality we have that the following diagram commutes:

$$\begin{array}{ccc} \mathbf{Nat}_2(W, \mathcal{B}(\lim F, F-)) & \xrightarrow{\lambda_{\lim F}} & \mathcal{B}(\lim F, \lim F) \\ {\scriptstyle \mathbf{Nat}_2(W,\mathcal{B}(g,F-))}\downarrow & & \downarrow{\scriptstyle -\circ g} \\ \mathbf{Nat}_2(W, \mathcal{B}(B, F-)) & \xrightarrow{\lambda_B} & \mathcal{B}(B, \lim F) \end{array}$$

Since $\lambda_{\lim F}(\alpha) = Id_{\lim F}$, we have that the path above equals g and the path below is given by the image

$$(B \xrightarrow{g} \lim F \xrightarrow{\alpha_A} FA)_{A \in \mathcal{A}}.$$

Thus we have

$$\alpha_A \circ g = f_A = \alpha_A \circ \lambda_B(\alpha^{(B,f)}), \quad \forall A \in \mathcal{A}.$$

So we have that the induced (2-)natural transformations are equal from which we conclude:

$$g = \lambda_B(\lambda_B^{-1}(g)) = \lambda_B(\alpha^{(B,f)}).$$

Conversely, assume $\lim F$ is the limit of F, i.e. we have an universal cone

$$\left(\lim F, \left(\lim F \xrightarrow{f_A} FA\right)_{A \in \mathcal{A}}\right).$$

By the bijective correspondance between the cones and natural transformations, we have for each $B \in \mathcal{B}$ a bijection

$$\mathbf{Nat}_2(\Delta, \mathcal{B}(B, F-)) \xrightarrow{\lambda_B} \mathcal{B}(B, \lim F), \quad B \in \mathcal{B}.$$

Indeed: Every morphism $f : B \to \lim F$ clearly defines a cone for F:

$$(B \xrightarrow{f} \lim F \xrightarrow{f_A} FA)_{A \in \mathcal{A}}.$$

And conversely, every cone $\left(B, \left(B \xrightarrow{g_A} FA\right)_{A \in \mathcal{A}}\right)$, defines a unique morphism $g : B \to \lim F$ by the universal property of $\lim F$. Thus it remains to show the naturality of λ_B in $B \in \mathcal{B}$, i.e. for each morphism $g : B \to \tilde{B}$, the following diagram should commute:

$$\begin{array}{ccc} \mathbf{Nat}_2(W, \mathcal{B}(\tilde{B}, F-)) & \xrightarrow{\lambda_{\tilde{B}}} & \mathcal{B}(\tilde{B}, \lim F) \\ {\scriptstyle \mathbf{Nat}_2(W,\mathcal{B}(g,F-))}\downarrow & & \downarrow{\scriptstyle -\circ g} \\ \mathbf{Nat}_2(W, \mathcal{B}(B, F-)) & \xrightarrow{\lambda_B} & \mathcal{B}(B, \lim F) \end{array}$$

Consider a cone $\left(\tilde{B} \xrightarrow{f_A} FA\right)_{A\in\mathcal{A}}$ for F and let $f : \tilde{B} \to \lim F$ be the unique morphism induced by the universal property of $\lim F$, i.e. $\lambda_{\tilde{B}}((B,(f_A)_A)) = f$. Then is the path above given by $f \circ g$. The path below is given by the unique morphism $B \to \lim F$ which corresponds with the cone

$$\left(B \xrightarrow{g} \tilde{B} \xrightarrow{f_A} FA\right)_{A\in\mathcal{A}}.$$

Thus by uniqueness we conclude the claim. □

This characterization allows us to perform a *twofold generalization*. Firstly, we can allow Δ to be replaced by any other functor, which is called a *weight functor*. Secondly, this isomorphism of sets (i.e. bijections) can be extended to an enriched setting, so in this case, we make this an isomorphism of categories:

Definition 6. *The* ***weighted 2-limit*** *of a 2-functor $F : \mathcal{A} \to \mathcal{B}$ weighted by a 2-functor $W : \mathcal{A} \to$ **Cat** is a 0-cell $\lim_W F \in \mathcal{B}$ such that there are isomorphisms of (1-)categories*

$$\lambda_B : \mathbf{Nat_2}(W, \mathcal{B}(B, F(-))) \to \mathcal{B}(B, \lim_W F),$$

which are natural in B.
The ***Weighted 2-colimit*** *of a 2-functor $F : \mathcal{A} \to \mathcal{B}$ weighted by a 2-functor $W : \mathcal{A}^{op} \to$ **Cat** is a 0-cell $colim_W F \in \mathcal{B}$ such that there are isomorphisms of (1-)categories*

$$\lambda_B : \mathbf{Nat_2}(W, \mathcal{B}(F(-), B)) \to \mathcal{B}(colim_W F, B),$$

which are natural in B.

Lemma 5. *If the weighted 2-limit (resp. 2-colimit) of F weighted by W exists, it is unique up to isomorphism. Consequently we write $\lim_W F$ for the limit.*

Proof. If L_1 and L_2 are both weighted limits, then we have for each object B isomorphisms $\mathcal{B}(B, L_1) \cong \mathcal{B}(B, L_2)$ from which $L_1 \cong L_2$ follows by the Yoneda lemma. The same argument holds for weighted colimits. □

In the sequel of this text, we will also write weighted limits for weighted 2-limits. In the rest of this section, we look at some examples and properties of weighted limits.

Remark 1. *The universal property of $\lim_W F$ consists of two parts because we have isomorphisms of categories. So we have that the morphisms (i.e. 1-cells) are given by natural transformations, this is what we call the* ***1-dimensional aspect****, but we also have that the 2-cells are given by modifications, this is what we call the* ***2-dimensional aspect****.*
Denote by ξ the natural transformation $W \to \mathcal{B}(\lim_W F, G-)$ which corresponds with $Id \in \mathcal{B}(\lim_W F, \lim_W F)$. The 1-dimensional aspect tells us then that we can write each natural transformation $\alpha : W \to \mathcal{B}(B, F-)$ as $\mathcal{B}(f, F-) \circ \xi$ for a unique $f \in \mathcal{B}(B, \lim_W F)$, indeed: By naturality of λ_B, we have that for each 0-cell B the following diagram commutes:

$$\begin{array}{ccc}\mathcal{B}(\lim_W F, \lim_W F) & \xrightarrow{\lambda^{-1}_{\lim_W F}} & \mathbf{Nat_2}(W, \mathcal{B}(\lim_W F, F(-))) \\ {\scriptstyle -\circ\lambda_B(\alpha)}\downarrow & & \downarrow{\scriptstyle \mathbf{Nat_2}(W,\mathcal{B}(\lambda_B(\alpha),F-))} \\ \mathcal{B}(B, \lim_W F) & \xrightarrow{\lambda^{-1}_B} & \mathbf{Nat_2}(W, \mathcal{B}(B, F(-)))\end{array}$$

So we indeed have:

$$\begin{aligned}\alpha = \lambda_B^{-1}(\lambda_B(\alpha)) &= \lambda_B^{-1}(Id_{\lim_W F} \circ \lambda_B(\alpha)) \\ &= \mathbf{Nat_2}(W, \mathcal{B}(\lambda_B(\alpha), F-))(\xi) \\ &= \mathcal{B}(\lambda_B(\alpha), F-) \circ \xi\end{aligned}$$

That $f := \lambda_B^{-1}(\alpha)$ indeed holds because by the same reasoning, we must have $\lambda_B(f) = \alpha$ which then concludes the claim since λ_B is an isomorphism.
By the same argument, the 2-dimensional aspect tells us that each modification $\theta : \rho_1 \to \rho_2$ (where ρ_i are natural transformations $W \to \mathcal{B}(B, F-)$) as $\mathcal{B}(\alpha, F-) \circ \xi$ for a unique 2-cell $\alpha : h_1 \to h_2$ (with h_i 1-cells $B \to \lim_W F$).

Definition 7. *Let $\star$ be the terminal category and consider the constant functor $W : \mathcal{A} \to \mathbf{Cat} : A \mapsto \star$. A limit weighted by such W is called a* ***conical limit****.*

Since the weight is trivial for a conical limit, it allows us to generalize the (ordinary) limits from 1-category theory:

Example 6. *("Examples of conical limits")*

- *Let $\mathcal{A}$ be the 2-category generated by 2 objects (denoted them by x, y) with no further relations. Let $X, Y \in \mathcal{B}$ be 0-cells. The* ***2-product*** *of X with Y (if it exists) is the conical limit of $F : \mathcal{A} \to \mathcal{B}$ which maps $x \mapsto X$ and $y \mapsto Y$. We now calculate its universal property:*
  A natural transformation $\alpha \in Fun_2(\mathcal{A}, \mathbf{Cat})(W, \mathcal{B}(B, F-))$ consists of 2 1-cells
  $$\alpha_x : W(x) \to \mathcal{B}(B, X), \alpha_y : W(y) \to \mathcal{B}(B, Y).$$
  Since W is trivial, both $W(x)$ and $W(y)$ are the terminal category, thus α_x and α_y are given by 1-cells $f_x : B \to X$ and $f_y : B \to Y$. Since there are no relations in $\mathcal{A}$, there are no naturality conditions between f_x and f_y, thus have that a natural transformation is given by morphisms $B \to X$ and $B \to Y$ with no relations between those.
  Now consider a modification $\Gamma : \alpha \to \beta$ (with α, β natural transformations $W \to \mathcal{B}(B, F-)$), so it consists of 2-cells $\Gamma_x : \alpha_x \to \beta_x$ and $\Gamma_y : \alpha_y \to \beta_y$, these are 2-cells in **Cat***, hence are natural transformations. Let f_x (resp. g_x, f_y, g_y) be the 1-cell corresponding to α_x (resp. $\beta_x, \alpha_y, \beta_y$). So Γ_x (resp. Γ_y) consists of only a 2-cell $f_x \to g_x$ (resp. $f_y \to g_y$), that is a morphism in $\mathcal{B}(B, X)$ (resp. $\mathcal{B}(B, Y)$). Since there are no relations between x and y, there are in particular no naturality conditions between Γ_x and Γ_y.*
  Thus all together we conclude:
  $$Fun_2(\mathcal{A}, \mathbf{Cat})(W, \mathcal{B}(B, F-)) \cong \mathcal{B}(B, X) \times \mathcal{B}(B, Y).$$

So if the conical limit $\lim_W F$ *exists, we have isomorphisms*

$$\mathcal{B}(B, \lim_W F) \cong \mathcal{B}(B, X) \times \mathcal{B}(B, Y),$$

which are natural in $B \in \mathcal{B}$.
Notice that the 1*-dimensional aspect of this limit means precisely that* $\lim_W F$ *is the (ordinary) limit in* $\mathcal{B}_0$ *(the underlying* 1*-category of* $\mathcal{B}$*), indeed: If we let* $B := \lim_W F$*, then corresponds* $Id \in \mathcal{B}(\lim_W F, \lim_W F)$ *with morphisms* $p : \lim_W F \to X, q : \lim_W F \to Y$ *and these are the morphisms which make* $\lim_W F$ *into the product because the* 1*-dimensional aspect (so the naturality of* B*) tells us that for each* B*, a morphism* $(f : B \to X, g : B \to Y)$ *is of the form* $(q, p) \circ h$ *for a unique morphism* $h : B \to \lim_W F$.

- *Let* $\mathcal{A}$ *be the* 2*-category generated by the following diagram:*

$$\begin{array}{ccc} & & y \\ & & \downarrow{\scriptstyle a} \\ x & \xrightarrow{b} & z \end{array}$$

The 2***-pullback*** *of* 1*-cells* $i \in \mathcal{B}(X, Z)$ *with* $j \in \mathcal{B}(Y, Z)$ *is the conical limit of* $F : \mathcal{A} \to \mathcal{B}$ *which is defined as:*

$$x \mapsto X, y \mapsto Y, z \mapsto Z, a \mapsto i, b \mapsto j.$$

We now calculate its universal property. For the same reason as with the 2-product, a natural transformation $\alpha : W \to \mathcal{B}(B, F-)$ *consists of* 3 *morphisms* $f_x : B \to X, f_y : B \to Y$ *and* $f_z : B \to Z$ *(these correspond with* α_x, α_y *and* α_z*). Since we have morphisms* $a : x \to z, b : y \to z$*, we have the following naturality conditions:*

$$\alpha_z \circ W(a) = i \circ \alpha_x, \alpha_z \circ W(b) = j \circ \alpha_y.$$

Since $W(a)$ *and* $W(b)$ *are identity functors on the terminal category, these equations mean precisely*

$$j \circ f_y = f_z = i \circ f_x.$$

Consider again modification $\Gamma : \alpha \to \beta$ *(with* α, β *natural transformations* $W \to \mathcal{B}(B, F-)$*). So* Γ *consists of natural transformations* $\Gamma_x : \alpha_x \to \beta_x, \Gamma_y : \alpha_y \to \beta_y, \Gamma_z : \alpha_z \to \beta_z$ *and (as in the case with the* 2*-product), they are completely determined by* 3 2*-cells (in* $\mathcal{B}$*) which we denote by:*

$$\gamma_x : f_x \to g_x, \gamma_y : f_y \to g_y, \gamma_z : f_z \to g_z.$$

Because we have morphisms $a : x \to z$ *and* $b : y \to z$*, we have the following naturality conditions:*

$$\Gamma_z \circ Id_{W(a)} = Id_i \circ \Gamma_x, \quad \Gamma_z \circ Id_{W(b)} = Id_j \circ \Gamma_y.$$

Because $W(a), W(b)$ *are identities, these equations mean precisely:*

$$i \circ \gamma_x = \gamma_z = j \circ \gamma_y.$$

So we conclude that the category $Fun_2(\mathcal{A}, \mathbf{Cat})(W, \mathcal{B}(B, F-))$ *consists of the following data:*

– *Objects:* $(f_x, f_y, f_z) \in \mathcal{B}(B, X) \times \mathcal{B}(B, Y) \times \mathcal{B}(B, Z)$ *such that* $j \circ f_y = f_z = i \circ f_x$.
– *Morphisms:* $(\gamma_x, \gamma_y, \gamma_z) \in \mathcal{B}(B, X)(f_x, g_x) \times \mathcal{B}(B, Y)(f_y, g_y) \times \mathcal{B}(B, Z)(f_z, g_z)$ *such that* $i \circ \gamma_x = \gamma_z = j \circ \gamma_y$.

So if the (weighted) limit exists, we have that the 1*-dimensional aspect means precisely that this limit is a pullback.*

Let $\mathcal{B}$ be a 2-category and $\mathcal{B}_0$ be its underlying 1-category. Example (6) illustrates that we can generalize the ordinary limits, but in general do not have that if an ordinary limit exists in $\mathcal{B}_0$, it is also the conical limit of the same diagram, i.e. the 1-dimensional aspect does not imply the 2-dimensional aspect:

Example 7. *Consider the* 2*-category* $\mathcal{B}$ *generated by the following diagram:*

$$0 \; \underset{f}{\overset{f}{\underset{\longrightarrow}{\overset{\longrightarrow}{\Downarrow \alpha}}}} \; 1 \ .$$

The underlying 1*-category is then just the arrow category, i.e. there are* 2 *objects* 0 *and* 1 *and one non-identity morphism* $f : 0 \to 1$*. The product* 1×1 *is just* 1 *(that this holds is immediate, but one can for example see it because* 1 *is terminal in* $\mathcal{B}_0$*). But if* 1 *would also be the* 2*-product, then it should satisfy (by example (6)):*

$$\mathcal{B}(0, 1) \cong \mathcal{B}(0, 1) \times \mathcal{B}(0, 1),$$

which is not true, indeed: $\mathcal{B}(0, 1)$ *is a category with* 1 *object and* 2 *morphisms (from that object to itself). Although* $\mathcal{B}(0, 1) \times \mathcal{B}(0, 1)$ *has only* 1 *object* (f, f)*, it has* 4 *morphisms* $(Id, Id), (Id, \alpha), (\alpha, Id), (\alpha, \alpha)$*, so there can't be an isomorphism.*

If we consider 1-categories as 2-categories, we have the following result:

Example 8. *Let* $\mathcal{B}$ *be a* 1*-category considered as a* 2*-category whose* 2*-cells are trivial. Let* $F : \mathcal{A} \to \mathcal{B}$ *be a functor. The (ordinary* 1*-)limit of* F *(if it exists) is the conical limit of* F*.*

Proof. By proposition (1), we have bijections

$$\lambda_B : \mathbf{Nat_2}(W, \mathcal{B}(B, F(-))) \to \mathcal{B}(B, \lim_W F),$$

which are natural in B. Thus we have to show that these bijections become isomorphisms of categories. But this is immediate since there are no non-trivial 2-cells in $\mathcal{B}(B, \lim_W F)$ and there are also no non-trivial modifications since a modification consists of 2-cells. So this clearly becomes an isomorphism of categories and since there are only identity 2-cells, the naturality at the level of the 2-cells is also clearly satisfied. □

Definition 8. *A* 2*-category* $\mathcal{B}$ *is* ***complete*** *(resp. finitely complete) if all weighted limits exists for all* $W : \mathcal{A} \to \mathbf{Cat}$ *and all* $F : \mathcal{A} \to \mathbf{B}$ *(resp. for all such* W *and* F *provided* $\mathcal{A}$ *is small).*

The proof of the following result can be found in ([6]):

Proposition 2. *Let $\mathcal{B}$ be a 1-category with trivial 2-cells. Then is $\mathcal{B}$ complete (in the 2-categorical sense) if $\mathcal{B}$ is complete in the 1-categorical sense, i.e. every weighted limit be computed as a conical limit.*

Example 9. *The weighted limit of a* **Cat**-*valued functor $F : \mathcal{A} \to \mathbf{Cat}$ (with weight $W : \mathcal{A} \to \mathbf{Cat}$) is given by $Fun_2(\mathcal{A}, \mathbf{Cat})(W, F)$. Consequently,* **Cat** *is complete.*

Proof. To show the claim, we have to show that there exist isomorphisms

$$\phi^{\mathcal{C}} : \mathbf{Cat}(\mathcal{C}, Fun_2(\mathcal{A}, \mathbf{Cat})(W, F)) \cong Fun_2(\mathcal{A}, \mathbf{Cat})(W, \mathbf{Cat}(\mathcal{C}, F-)),$$

natural in $\mathcal{C}$ (a small 1-category).
Let $G : \mathcal{C} \to Fun_2(\mathcal{A}, \mathbf{Cat})(W, F)$ be a functor. We have to define a 2-natural transformation $\phi^{\mathcal{C}}(G) : W \implies \mathbf{Cat}(\mathcal{C}, F-)$. Let $X \in \mathcal{A}$, define

$$\phi^{\mathcal{C}}(G)_X : W(X) \to \mathbf{Cat}(\mathcal{C}, FX),$$

as follows: For each $x \in W(X)$, define

$$\phi^{\mathcal{C}}(G)_X(x) : \mathcal{C} \to FX : c \mapsto G(c)_X(x), (g : c \to \tilde{c}) \mapsto G(g)_X(x).$$

Since G is a functor, we have that $\phi^{\mathcal{C}}(G)_X(x) : \mathcal{C} \to FX$ is a functor.
And for each morphism $f : x \to y \in W(X)$, define

$$\phi^{\mathcal{C}}(G)_X(f) : \phi^{\mathcal{C}}(G)_X(x) = G(-)_X(x) \to \phi^{\mathcal{C}}(G)_X(y) = G(-)_X(y),$$

by $\left(\phi^{\mathcal{C}}(G)_X(f)\right)_c := G(c)_X(f)$. This is a natural transformation, indeed: We have to show for each morphism $g : c \to d \in \mathcal{C}$, that the following diagram commutes:

$$\begin{array}{ccc} G(c)_X(x) & \xrightarrow{G(c)_X(f)} & G(c)_X(y) \\ {\scriptstyle G(g)_X(x)}\downarrow & & \downarrow{\scriptstyle G(g)_X(y)} \\ G(d)_X(x) & \xrightarrow{G(d)_X(f)} & G(d)_X(y) \end{array}$$

But $G(g)$ is a modification, thus $G(g)_X$ is a 2-cell in **Cat**, i.e. a natural transformation. Therefore the diagram commutes by naturality of $G(g)_X$.
We now show that $\phi^{\mathcal{C}}(G)$ is indeed a 2-natural transformation, i.e. $\phi^{\mathcal{C}}(G)_X$ is natural in X. So we have to show that for each 1-cell $h \in \mathcal{A}(X, Y)$ and each 2-cell $\beta \in \mathcal{A}(X, Y)(h, i)$, the following diagrams commute:

$$\begin{array}{ccc} W(X) & \xrightarrow{\phi^{\mathcal{C}}(G)_X} & \mathbf{Cat}(\mathcal{C}, FX) \\ {\scriptstyle W(h)}\downarrow & & \downarrow{\scriptstyle \mathbf{Cat}(\mathcal{C}, Fh)} \\ W(Y) & \xrightarrow{\phi^{\mathcal{C}}(G)_Y} & \mathbf{Cat}(\mathcal{C}, FY) \end{array}, \qquad \begin{array}{ccc} W(X) & \xrightarrow{\phi^{\mathcal{C}}(G)_X} & \mathbf{Cat}(\mathcal{C}, FX) \\ {\scriptstyle W(h)}\downarrow \overset{W(\beta)}{\Longrightarrow} \downarrow{\scriptstyle W(i)} & & {\scriptstyle Fh\circ-}\downarrow \overset{F\beta\circ-}{\Longrightarrow} \downarrow{\scriptstyle Fi\circ-} \\ W(Y) & \xrightarrow{\phi^{\mathcal{C}}(G)_Y} & \mathbf{Cat}(\mathcal{C}, FY) \end{array}$$

Unwrapping the definitions, we have:

$$\begin{aligned} \phi^{\mathcal{C}}(G)_Y(W(h)(-)) :& \mathcal{C} \to FY : \\ & c \mapsto G(c)_Y(W(h)(-)), \\ \mathbf{Cat}(\mathcal{C}, Fh)(\phi^{\mathcal{C}}(G)_X(-)) :& \mathcal{C} \to FY : \\ & c \mapsto F(h)\left(G(c)_X(-)\right). \end{aligned}$$

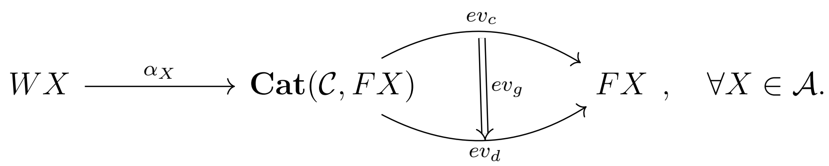
And the same for h replaced by β. Since $G(c)$ is a 2-natural transformation, we have (by naturality) that these are equal (for both h and β). Thus we indeed have that $\phi^{\mathcal{C}}$ is well-defined on objects (i.e. functors). We now show that this gives a bijection on the objects. We first show that $\phi^{\mathcal{C}}$ is injective on objects: Let $G, H : \mathcal{C} \to Fun_2(\mathcal{A}, \mathbf{Cat})(W, F)$ be functors such that $\phi^{\mathcal{C}}(G) = \phi^{\mathcal{C}}(H)$. Thus, for each $X \in \mathcal{A}$, we have (by definition of $\phi^{\mathcal{C}}$):

$$\begin{array}{rcl} \forall x \in W(X) & : & G(-)_X(x) =: \phi^{\mathcal{C}}(G)_X(x) = \phi^{\mathcal{C}}(H)_X(x) := H(-)_X(x) \\ \forall f \in W(X)(x,y) : \forall c \in \mathcal{C} & : & G(c)_X(f) =: \left(\phi^{\mathcal{C}}(G)_X(f)\right)_c = \left(\phi^{\mathcal{C}}(H)_X(f)\right)_c := H(c)_X(f) \end{array}$$

Which shows $G = H$.
Now consider a 2-natural transformation $\alpha : W \implies \mathbf{Cat}(\mathcal{C}, F-)$. Define a functor $G : \mathcal{C} \to Fun_2(\mathcal{A}, \mathbf{Cat})(W, F)$ by mapping an object $c \in \mathcal{C}$ to the 2-natural transformation $G(c)$ which is object-wise given by:

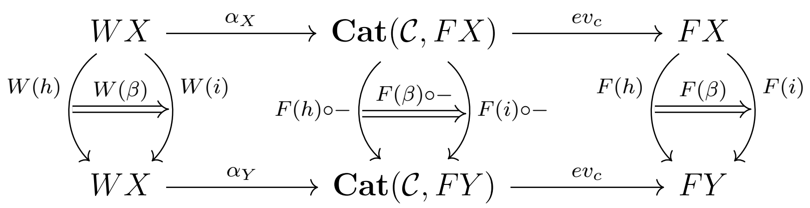
$$WX \xrightarrow{\alpha_X} \mathbf{Cat}(\mathcal{C}, FX) \xrightarrow{ev_c} FX, \quad \forall X \in \mathcal{A},$$

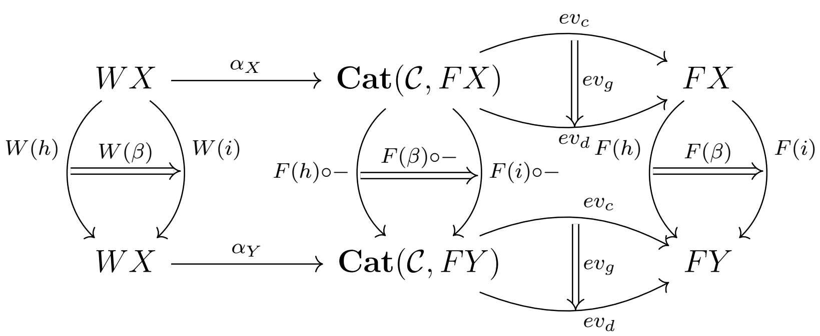
and each morphism $g : c \to d$ is mapped to the modification $G(g)$ which is object-wise given by:



That $G(c)$ (resp. $G(g)$) is a 2-natural transformation (resp. modification) follows because for each 1-cell $h \in \mathcal{A}(X,Y)$ and for each 2-cell $\beta \in \mathcal{A}(X,Y)(h,i)$, the following diagrams commute:

$$\begin{array}{ccccc} WX & \xrightarrow{\alpha_X} & \mathbf{Cat}(\mathcal{C}, FX) & \xrightarrow{ev_c} & FX \\ {\scriptstyle W(h)}\downarrow & & {\scriptstyle Fh\circ-}\downarrow & & \downarrow{\scriptstyle Fh} \\ WX & \xrightarrow{\alpha_Y} & \mathbf{Cat}(\mathcal{C}, FY) & \xrightarrow{ev_c} & FY \end{array}$$





Indeed: The left squares commute because α is a 2-natural transformation and the right squares clearly commute because evaluating after the composition (with Fh

resp. $F\beta$) is the same as first evaluating and then applying it to Fh (resp. $F\beta$). So $G : \mathcal{C} \to Fun_2(\mathcal{A}, \mathbf{Cat})$ is a well-defined, it only remains to check functoriality of G. That G preserves composition means that if $g : c \to d$ and $f : d \to e$ are morphisms (in $\mathcal{C}$), then $G(f \circ g) = G(f) \circ G(g)$ (where $\circ$ is the vertical composition of modifications). By definition of G, this equality means that for each $X \in \mathcal{A}$, we should have

$$\alpha_X(-)(f \circ g) = \alpha_X(-)(f) \circ \alpha_X(-)(g).$$

For $x \in W(X)$, $\alpha_X(x) : \mathcal{C} \to FX$ is a functor, hence $\alpha_X(x)(f) \circ \alpha_X(x)(g) = \alpha_X(x)(f \circ g)$. For $k \in W(X)(x, y)$, we have that $\alpha_X(k) : \alpha_X(x) \to \alpha_X(y)$ is a natural transformation and $\alpha_X(k)_c$ is a functor for every $c \in \mathcal{C}$ from which we conclude $\alpha_X(-)(f \circ g) = \alpha_X(-)(f) \circ \alpha_X(-)(g)$.
The definition of $\phi^{\mathcal{C}}$ on modifications and the fully faithfulness is similar. □

Corollary 1. *Let $\mathcal{A}$ be a small 2-category, then is $Fun_2(\mathcal{A}, \mathbf{Cat})$ complete where the limits are formed objectwise.*

Proof. Let $F : \mathcal{B} \to Fun_2(\mathcal{A}, \mathbf{Cat})$ be a 2-functor and consider a weight 2-functor $W : \mathcal{B} \to \mathbf{Cat}$. To show that the 2-functor category has all weighted limits, we have to show that there exists a 2-functor $\lim_W F : \mathcal{A} \to \mathbf{Cat}$ such that there are isomorphisms

$$\tilde{\lambda}_G : Nat_2(W, Fun_2(\mathcal{A}, \mathbf{Cat})(G, F-)) \to Fun_2(\mathcal{A}, \mathbf{Cat})(G, \lim_W F),$$

which are natural in $G \in Fun_2(\mathcal{A}, \mathbf{Cat})$.
Let $A \in \mathcal{A}$ be a 0-cell and define for each such A the (small) category L_A which is the limit of the 2-functor induced by

$$F(-)(A) : \mathcal{B} \to \mathbf{Cat} : B \mapsto F(B)(A),$$

weighted by W. By the previous example, we know that $\mathbf{Cat}$ is complete, thus this weighted limit L_A indeed exists for each $A \in \mathcal{A}$. So for each $A \in \mathcal{A}$, we have isomorphisms

$$\lambda_X^{(A)} : Nat_2(W, \mathbf{Cat}(X, F(-)(A))) \to \mathbf{Cat}(X, L_A),$$

which is natural in $X \in \mathbf{Cat}$.
We now claim that $\lim_W F$ is given objectwise, i.e.

$$\lim_W F : \mathcal{A} \to \mathbf{Cat} : A \mapsto L_A.$$

We now make $\lim_W F$ into a 2-functor as follows: Let $f \in \mathcal{A}(\tilde{A}, A)$ be a 1-cell. We want to map this to a functor $L_f : L_{\tilde{A}} \to L_A$. Since L_A is the weighted limit, we have the following isomorphism:

$$\lambda_{L_{\tilde{A}}}^{(A)} : \mathbf{Nat}_2(W, \mathbf{Cat}(L_{\tilde{A}}, F(-)(A))) \to \mathbf{Cat}(L_{\tilde{A}}, L_A).$$

So we define L_f corresponding to the following 2-natural transformation:

$$W(B) \xrightarrow{\left(\lambda_{L_{\tilde{A}}}^{(\tilde{A})}\right)^{-1}(Id_{L_{\tilde{A}}})} \mathbf{Cat}(L_{\tilde{A}}, F(B)(\tilde{A})) \xrightarrow{F(B)(f)\circ -} \mathbf{Cat}(L_{\tilde{A}}, F(B)(A)),$$

which we denote by $N_f \equiv (N_f(B))_{B\in\mathcal{B}}$, i.e. L_f is the functor

$$L_f(-) = \lambda^{(A)}_{L_{\tilde{A}}}(N_f) = \lambda^{(A)}_{L_{\tilde{A}}}\left(\left\{F(B)(f)\circ\left(\left(\left(\lambda^{(\tilde{A})}_{L_{\tilde{A}}}\right)^{-1}\left(Id_{L_{\tilde{A}}}\right)\right)(-)\right)\right\}_{B\in\mathcal{B}}\right).$$

It remains to define $\lim_W F$ on 2-cells, but this is exactly the same as defining L_f because $\lambda^{(A)}_{L_{\tilde{A}}}$ gives a bijection not only between the 2-natural transformations and the functors, but also between the modifications and the $(1-)$natural transformations. That $\lim_W$ preserves the composition follows from the naturality of $\lambda_{L_{A_1}}$, indeed: Let $A_1 \xrightarrow{f} A_2 \xrightarrow{g} A_3$ be morphisms in $\mathcal{A}$. By naturality the following diagram commutes:

$$\begin{array}{ccc}
Nat_2(W, \mathbf{Cat}(L_{A_1}, F(-)(A_2))) & \xrightarrow{\lambda^{(A_2)}_{L_{A_1}}} & \mathbf{Cat}(L_{A_1}, L_{A_2}) \\
{\scriptstyle \mathbf{Cat}(L_{A_2},F(-)(g))}\downarrow & & \downarrow{\scriptstyle g\circ -} \\
Nat_2(W, \mathbf{Cat}(L_{A_1}, F(-)(A_3))) & \xrightarrow{\lambda^{(A_3)}_{L_{A_1}}} & \mathbf{Cat}(L_{A_1}, L_{A_3})
\end{array}$$

Thus applying this to N_f, we have:

$$\begin{aligned}
L_g \circ L_f &= L_g \circ \lambda_{L^{(A_2)}_{A_1}}(N_f) \\
&= \lambda_{L^{(A_3)}_{A_1}}\left(W(B) \xrightarrow{N_f(B)} \mathbf{Cat}(L_{A_1}, F(B)(A_2)) \xrightarrow{F(B)(g)\circ -} \mathbf{Cat}(L_{A_1}, F(B)(A_3))\right)_{B\in\mathcal{B}} \\
&= \lambda_{L^{(A_3)}_{A_1}}\left((F(B)(g)\circ -)\circ(F(B)(f)\circ -)\circ\lambda_{(L^{(A_1)}_{A_1})^{-1}(Id_{L_{A_1}})}\right)_{B\in\mathcal{B}} \\
&= \lambda_{L^{(A_3)}_{A_1}}\left((F(B)(g\circ f)\circ -)\circ\lambda_{(L^{(A_1)}_{A_1})^{-1}(Id_{L_{A_1}})}\right)_{B\in\mathcal{B}} \\
&= \lambda_{L^{(A_3)}_{A_1}}(N_{g\circ f}) = L_{g\circ f}
\end{aligned}$$

where the third equality holds by definition of $N_f(-)$ and the fourth holds by functoriality of $F(B)$ (for each $B \in \mathcal{B}$). The same computation holds for 2-cells which shows that the composition is preserved, i.e. the following diagram commutes:

$$\begin{array}{ccc}
\mathcal{A}(A_1, A_2) \times \mathcal{A}(A_2, A_3) & \xrightarrow{c_{A_1,A_2,A_3}} & \mathcal{A}(A_1, A_3) \\
{\scriptstyle (\lim_W F)_{A_1,A_2}\times(\lim_W F)_{A_2,A_3}}\downarrow & & \downarrow{\scriptstyle (\lim_W F)_{A_1,A_3}} \\
\mathbf{Cat}(L_{A_1}, L_{A_2}) \times \mathbf{Cat}(L_{A_2}, L_{A_3}) & \xrightarrow{c_{L_{A_1},L_{A_2},L_{A_3}}} & \mathbf{Cat}(L_{A_1}, L_{A_3})
\end{array}$$

We now define $\tilde{\lambda}_G$. Let $\alpha : W \to Fun_2(\mathcal{A}, \mathbf{Cat})(G, F-)$ be a 2-natural transformation. We have to define a 2-natural transformation $\tilde{\lambda}_G(\alpha) : G \to \lim_W F$. Thus for each $A \in \mathcal{A}$, we need a functor $\tilde{\lambda}_G(\alpha)_A : GA \to L_A$. Since L_A is a weighted limit (in **Cat**), this corresponds with a 2-natural transformation $\beta : W \to \mathbf{Cat}(GA, F(-)(A))$. Define $\beta_B := ev_A \circ \alpha_B$. Since α is a 2-natural transformation, so is β. Thus $\tilde{\lambda}_G(\alpha)_A := \lambda^{(A)}_{GA}(\beta)$. For a modification $\Gamma : \alpha \to \tilde{\alpha}$, we can do the same, i.e. define $\tilde{\lambda}_G(\Gamma)_A$ as the image of the modification given by $ev_A \circ \Gamma_B$ under $\lambda^{(A)}_{GA}$. Since $\lambda^{(A)}_{GA}$ is an isomorphism, we have that $\tilde{\lambda}_G$ is an isomorphism. So it remains to check the naturality in G. Since 2-natural transformations and modifications are given object-wise, this naturality has to be checked object-wise and this holds because $\lambda_X^{(A)}$ is natural in X (so in particular GA). $\square$

Lemma 6. *Let $\mathcal{A}$ be a small 2-category. Consider the Yoneda embedding*

$$\mathcal{Y} : \mathcal{A}^{op} \to Fun_2(\mathcal{A}, \mathbf{Cat}) : A \mapsto \mathcal{A}(A, -).$$

Then we have $F \cong colim_F Y$ for every 2-functor $F : \mathcal{A} \to \mathbf{Cat}$.

Proof. Let $G : \mathcal{A} \to \mathbf{Cat}$ be a 2-functor. By Yoneda we have for each 0-cell $A \in \mathcal{A}$ an isomorphism

$$\phi_A : Fun_2(\mathcal{A}, \mathbf{Cat})(\mathcal{A}(A, -), G) \cong GA,$$

Since $\mathcal{Y}A = \mathcal{A}(A, -)$, this induces an isomorphism (in $\mathbf{Cat}$):

$$\lambda : Nat_2(F, Fun_2(\mathcal{A}, \mathbf{Cat})(\mathcal{Y}-, G)) \cong Fun_2(\mathcal{A}, \mathbf{Cat})(F, G).$$

Indeed: Consider a 2-natural transformation $\alpha : F \to Fun_2(\mathcal{A}, \mathbf{Cat})(\mathcal{Y}-, G)$. This is mapped to the 2-natural transformation $\lambda(\alpha) : F \to G$ defined by

$$\lambda(\alpha)_A := \phi_A \circ \alpha_A : FA \to GA.$$

This is natural in A because α and λ are natural 2-transformations, indeed: If $f \in \mathcal{A}(A, B)$ is a 1-cell, then by naturality of α and ϕ, we have that the following diagram commutes:

$$\begin{array}{ccccc} FA & \xrightarrow{\alpha_A} & Fun_2(\mathcal{A}, \mathbf{Cat})(\mathcal{A}(A, -), G) & \xrightarrow{\phi_A} & GA \\ \downarrow{\scriptstyle F(f)} & & \downarrow{\scriptstyle Fun_2(\mathcal{A}, \mathbf{Cat})(\mathcal{A}(f,-),G)} & & \downarrow{\scriptstyle G(f)} \\ FB & \xrightarrow{\alpha_B} & Fun_2(\mathcal{A}, \mathbf{Cat})(\mathcal{A}(B, -), G) & \xrightarrow{\phi_B} & GB \end{array}$$

So $\lambda(\alpha)$ is indeed a 2-natural transformation.
Now consider a modification $\Gamma : \alpha^{(1)} \to \alpha^{(2)}$. This is mapped to the modification $\lambda(\Gamma)$ defined by

$$\lambda(\Gamma)_A := Id_{\phi_A} \circ \Gamma_A : \phi_A \circ \alpha_A^{(1)} \to \phi_A \circ \alpha_A^{(2)},$$

where Id_{ϕ_A} is the identity 2-cell $\phi_A \to \phi_A$. $\lambda(\Gamma)$ is indeed a modification because for each 2-cell $\gamma : f \to g : A \to B$ we have:

$$\begin{aligned} Id_{\phi_B} \circ \Gamma_B \circ F(\gamma) &= Id_{\phi_B} \circ Fun_2(\mathcal{A}, \mathbf{Cat})(\mathcal{A}(\gamma, -), G) \circ \Gamma_A \\ &= G(\gamma) \circ Id_{\phi_A} \circ \Gamma_A \end{aligned}$$

where the first (resp. second) equality holds because Γ (resp. Id_ϕ) is a modification. We now show that λ is functorial. That λ preserves the identity 2-cell is immediate because the (horizontal) composition of an identity 2-cell with the identity 2-cell is again an identity 2-cell. That λ preserves composition means that for modifications $\Gamma : \alpha^{(1)} \to \alpha^{(2)}, \tilde{\Gamma} : \alpha^{(2)} \to \alpha^{(3)}$ (both from $F \to Fun_2(\mathcal{A}, \mathbf{Cat})(Y-, G)$), we should have $\lambda(\tilde{\Gamma} \circ \Gamma) = \lambda(\tilde{\Gamma}) \circ \lambda(\Gamma)$, i.e. for each 0-cell $A \in \mathcal{A}$ we should have

$$Id_{\phi_A} \circ (\tilde{\Gamma} \circ \Gamma)_A = \lambda(\tilde{\Gamma} \circ \Gamma)_A = \lambda(\tilde{\Gamma})_A \circ \lambda(\Gamma)_A = (Id_{\phi_A} \circ \tilde{\Gamma}_A) \circ (Id_{\phi_A} \circ \Gamma_A).$$

Since $Id_{\phi_A} = Id_{\phi_A} \circ Id_{\phi_A}$, this equality holds precisely by the **interchange law** (1.1).

We now argue that λ is an isomorphism, i.e. it is bijective on objects and fully faithful, but this follows immediately since for each 0-cell $A \in \mathcal{A}$, we compose with an isomorphism ϕ_A (resp. Id_{Γ_A}). The only thing what needs to be noticed is that if α (resp. Γ) is a 2-natural transformation (resp. modification), then defines $\phi_A^{-1} \circ \alpha_A$ (resp. $Id_{\phi_A^{-1}} \circ \Gamma_A$) a 2-natural transformation (resp. modification) which shows that λ is surjective on objects (resp. full), but this is the same computation as before. We clearly have that λ is injective on objects, indeed: For each 2-natural transformations α, β we have that if $\lambda(\alpha) = \lambda(\beta)$, then we have for each A that

$$\phi_A \circ \alpha_A = \lambda(\alpha)_A = \lambda(\beta)_A = \phi_A \circ \beta_A,$$

but α_A is an isomorphism, hence we conclude for each A that $\alpha_A = \beta$ which shows the injectivity on objects. Faithfulness is shown in a similar way.
So we have that λ is an isomorphism, it only remains to prove that it is natural in G but this follows immediate by the naturality of ϕ_A (given by the Yoneda lemma) since the naturality has to be checked object-wise. □

We have seen an important class of weighted limits, the conical limits. We now introduce an important (non-conical)-weighted limit, called the cotensor:

Definition 9. *Let $\mathcal{A}$ be a 2-category, $A \in \mathcal{A}$, $V \in \mathbf{Cat}$. The* ***cotensor*** *of V and A is an object $V \pitchfork A$ such that for $B \in \mathcal{A}$, there is an isomorphism (of 1-categories)*

$$\mathcal{A}(B, V \pitchfork A) \cong \mathbf{Cat}(V, \mathcal{A}(B, A)),$$

natural in B. We call $\mathcal{A}$ ***cotensored over*** *V if the cotensors $V \pitchfork A$ exists for all $A \in \mathcal{A}$.*

Remark 2. *Let $\mathcal{C}$ be the 2-category generated by a single object $\star$. Notice that the cotensor of $V \in \mathbf{Cat}$ and $A \in \mathcal{A}$ is the limit of*

$$F : \mathcal{C} \to \mathcal{A} : \star \mapsto A,$$

which is weighted by

$$W : \mathcal{C} \to \mathbf{Cat} : \star \mapsto V.$$

Indeed, a natural transformation $W \to \mathcal{A}(B, F-)$ is given by a single 1-cell $V \to \mathcal{A}(B, A)$, which gives us the following isomorphism:

$$\mathcal{A}(B, V \pitchfork A) \cong Nat_2(W, \mathcal{A}(B, F-)) \cong \mathbf{Cat}(V, \mathcal{A}(B, A)).$$

The importance of the cotensor becomes clear when taking the cotensor of the **arrow 2-category 2** (the 2-category generated by the diagram $\cdot_1 \to \cdot_2$). If the cotensor (over **2**) exists, we have an isomorphism

$$\mathcal{A}(B, \mathbf{2} \pitchfork A) \cong \mathbf{Cat}(\mathbf{2}, \mathcal{A}(B, A)).$$

But a functor $F : \mathbf{2} \to \mathcal{A}(B, A)$ consists of 2 1-cells $f, g : B \to A$ and a 2-cell $\alpha : f \to g$. So we can work with 2-cells as they were 1-cells in $\mathcal{A}$ from B to $2 \pitchfork A$. Therefore we can sometimes reduce the amount of work by only showing things for the 1-cells. An illustration of this concept is used in the section of sheafification.
Another useful theorem we have (which we will not use in this thesis, hence not proven), is the following:

Theorem 2. *Let $\mathcal{A}$ be a 2-category. Then is $\mathcal{A}$ finitely 2-complete if and only if it has all 2-products, 2-equalizers[1] and all cotensors with 2.*

1.4 Acute and chronic arrows

Important classes of morphisms in an ordinary category are those of the mono -and epimorphisms. The reason for this is that a lot of *nice* categories have an epi-mono factorization. In the 1-category **Cat**, a well-known factorization of a functor is that of the functors which are injective on objects and fully faithfull one the one side, and on the other side the essentially surjective (that is on objects) functors. In a 2-category $\mathcal{K}$, we call a 1-cell *chronic* if it is representable injective on objects and fully faithfull and the essentially surjective functors correspond with the so-called *acute 1-cells*.

Definition 10. *An arrow $m \in \mathcal{K}(X, Y)$ is* ***chronic*** *if for all $K \in \mathcal{K}$, the functor*

$$\mathcal{K}(K, m) : \mathcal{K}(K, X) \to \mathcal{K}(K, Y),$$

is injective on objects and fully faithfull.

Notice that injective on objects (for all K) means that m is a monomorphism.

Example 10. *In $\mathcal{K} =$ **Cat**, the chronic arrows are the fully faithfull functors which are moreover injective on objects.*

Proof. The monomorphisms in **Cat** are those functors which are injective on objects and morphisms.
Let $F : \mathcal{A} \to \mathcal{B}$ be a functor. We have to show that F is fully faithfull if and only if $\mathbf{Cat}(\mathcal{C}, -) \circ F$ is fully faithfull for all $\mathcal{C}$.
Let $\alpha, \beta : G \implies H : \mathcal{C} \to \mathcal{A}$ be natural transformations such that $F \circ \alpha = F \circ \beta$. So for each $C \in \mathcal{C}$ we have

$$F\left(G(C) \xrightarrow{\alpha_C} H(C)\right) = F\left(G(C) \xrightarrow{\beta_C} H(C)\right).$$

So by faithfullness of F we conclude that for each $C \in \mathcal{C}$, $\alpha_C = \beta_C$, so we conclude $\alpha = \beta$.
Assume F is full. Let $\alpha : FG \implies FH : \mathcal{C} \to \mathcal{B}$ be a natural transformation. So for each $C \in \mathcal{C}$ we have morphisms $\alpha_C : FG(C) \to FH(C)$ (in $\mathcal{B}$), so by fullness of F, there exists for each $C \in \mathcal{C}$ a morphism $\beta_C : G(C) \to H(C)$ (in $\mathcal{A}$) such that $F(\beta_C) = \alpha_C$. We can conclude fullness of $\mathbf{Cat}(\mathcal{C}, -) \circ F$ if $(\beta_C)_{C \in \mathcal{C}}$ forms a natural transformation, i.e. is natural in C, this follows from the naturality of α and faithfullness of F from the following computation:

$$\begin{aligned} F(\beta_{C_1} \circ G(f)) &= F(\beta_{C_1}) \circ FG(f)) \\ &= \alpha_{C_1} \circ FG(f) \\ &= FH(f) \circ \alpha_{C_2} \\ &= FH(f) \circ F(\beta_{C_2}) \\ &= F(H(f) \circ \beta_{C_2}) \end{aligned}$$

[1]The 2-equalizer is the conical limit of the 2-functor which corresponds with the usual diagram of the equalizer, i.e. just two 1-cells with the same domain and codomain.

where $f \in \mathcal{C}(C_2, C_1)$.
The converse follows from the fully faithfulness of $\mathbf{Cat}(\star, -) \circ F$ where $\star$ is the terminal category because $\mathbf{Cat}(\star, -)$ induces an isomorphism of categories between $\mathcal{A}$ and $\mathbf{Cat}(\star, \mathcal{A})$. To spell this out: A functor $\tilde{A} : \star \to \mathcal{A}$ is given by an object $A \in \mathcal{A}$ and a natural transformation between such functors $\tilde{\alpha} : \tilde{A}_1 \to \tilde{A}_2$ is given by a morphism $\alpha : A_1 \to A_2$ (between the corresponding objects) and the functor $\mathbf{Cat}(\star, -) \circ F$ applied to a natural transformation $\tilde{\alpha}$ is the same as applying F to the morphism α. □

Lemma 7. *The composition of chronic arrows is chronic and the pullback of a chronic arrow is again chronic.*

Proof. That the composition of chronics is chronic is clear because the composition of pullback squares is a pullbacksquare. Let $m \in \mathcal{K}(X, Y)$ be chronic and $f \in \mathcal{K}(Z, Y)$ arbitrary and consider the pullback $m^\star(f)$ of m along f, i.e. the following diagram is a pullback square:

$$\begin{array}{ccc} W & \xrightarrow{f^\star(m)} & X \\ \downarrow{\scriptstyle m^\star(f)} & & \downarrow{\scriptstyle m} \\ Z & \xrightarrow{f} & Y \end{array}$$

That $\mathcal{K}(K, m)$ is injective on objects for all K means that m is a monomorphism, since $m^\star(f)$ is the pullback of m, $\mathcal{K}(K, m^\star(f))$ is therefore also injective on objects for all K.
That $\mathcal{K}(K, m^\star(f))$ is fully faithfull follows because in $\mathbf{Cat}$ (considered as a 1-category) we have a weak factorisation system $(\mathcal{L}, \mathcal{R})$ with $\mathcal{L}$ consisting of those functors which are bijective on objects and $\mathcal{R}$ consistings of the fully faithfull functors and for general weak factorisation systems, $\mathcal{R}$ is closed under pullback. □

Lemma 8. *Let $J : \mathcal{A} \to \mathbf{Cat}$ and $S, T : \mathcal{A} \to \mathcal{K}$ be 2-functors and $\theta : S \to T$ a 2-natural transformation. If $\mathcal{K}$ admits J-indexed limits* $\lim(J, S)$ *and* $\lim(J, T)$ *and if each component of θ is chronic, then so is*

$$\lim(J, \theta) : \lim(J, S) \to \lim(J, T).$$

Proof. Recall that the object $\lim(J, S)$ is defined to be the unique object such that we have an isomorphism of categories

$$\lambda_X^{S,J} : \mathcal{K}(X, \lim(J, S)) \cong [\mathcal{A}, \mathbf{Cat}](J, \mathcal{K}(X, S(-))),$$

natural in X. The induced morphism $\lim(J, \theta)$ is given by

$$\lim(J, \theta) = \left(\lambda^{T,J}_{\lim_J S}\right)^{-1} (\mathcal{K}(\lim_J S, \theta) \cdot \lambda^{S,J}_{\lim_J S}(Id_{\lim(J,S)})).$$

Assume that $\lim(J, \theta) \circ f = \lim(J, \theta) \circ g$ for $f, g \in \mathcal{K}(X, \lim(J, S))$. The naturality of X means

$$\begin{array}{ccc} \mathcal{K}(\lim_J S, \lim_J T) & \xrightarrow{\lambda^{T,J}_{\lim_J S}} & [\mathcal{A}, \mathbf{Cat}]\,(J, \mathcal{K}(\lim_J S, T)) \\ \downarrow{\scriptstyle -\circ f} & & \downarrow{\scriptstyle \mathcal{K}(f,T)\circ -} \\ \mathcal{K}(X, \lim_J T) & \xrightarrow{\lambda^{T,J}_X} & [\mathcal{A}, \mathbf{Cat}]\,(J, \mathcal{K}(X, T)) \end{array}$$

Thus

$$\begin{aligned}
\lambda_X^{T,J}(\lim_J \theta \circ f) &= \mathcal{K}(f,T) \circ \lambda_{\lim_J S}^{T,J}(\lim_J \theta) \\
&= \mathcal{K}(f,T) \circ \mathcal{K}(\lim_J S, \theta) \circ \lambda_{\lim_J S}^{S,J}(Id_{\lim_J S}) \\
&= \mathcal{K}(\lim_J S, \theta) \circ \mathcal{K}(f,T) \circ \lambda_{\lim_J S}^{S,J}(Id_{\lim_J S})
\end{aligned}$$

So $\lambda_X^{T,J}(\lim_J \theta \circ f)$ is a natural transformation $J \implies \mathcal{K}(X, T(-))$ which is componentwise given by

$$J(A) \xrightarrow{\lambda_{\lim_J S}^{S,J}(Id_{\lim_J S}(A))} \mathcal{K}(\lim_J S, S(A)) \xrightarrow{\theta_A \circ -} \mathcal{K}(\lim_J S, T(A)) \xrightarrow{-\circ f} \mathcal{K}(X, T(A)).$$

So from $\lim(J,\theta) \circ f = \lim(J,\theta) \circ g$, we get

$$\theta_A \circ \lambda_{\lim_J S}^{S,J}(Id_{\lim_J S})(A) \circ f = \theta_A \circ \lambda_{\lim_J S}^{S,J}(Id_{\lim_J S})(A) \circ g.$$

So from chronicness of θ_A, we get

$$\lambda_{\lim_J S}^{S,J}(Id_{\lim_J S})(A) \circ f = \lambda_{\lim_J S}^{S,J}(Id_{\lim_J S})(A) \circ g.$$

As this holds for all A, we get

$$\mathcal{K}(f,T) \circ \lambda_{\lim_J S}^{S,J}(Id_{\lim_J S}) = \mathcal{K}(g,T) \circ \lambda_{\lim_J S}^{S,J}(Id_{\lim_J S}).$$

Again applying the naturality of X, this equation is equivalent to

$$\lambda_X^{J,S}(Id_{\lim_J S} \circ f) = \lambda_X^{J,S}(Id_{\lim_J S} \circ g),$$

which shows that $f = g$ and thus we conclude that $\mathcal{K}(X, \lim(J,\theta))$ is injective on objects.
That $\mathcal{K}(X, \lim(J,\theta))$ is faithfull follows from the same computation and to show the fullness, we use the same argument to show that we can describe each modification in a particular form from which it then follows. □

Definition 11. *An arrow $e \in \mathcal{K}(A,B)$ is **acute** if for all chronics $m : X \to Y$, the following diagram is a pullback square (in **Cat**):*

$$\begin{array}{ccc}
\mathcal{K}(B,X) & \xrightarrow{\mathcal{K}(B,m)} & \mathcal{K}(B,Y) \\
{\scriptstyle \mathcal{K}(e,X)}\downarrow & & \downarrow{\scriptstyle \mathcal{K}(e,Y)} \\
\mathcal{K}(A,X) & \xrightarrow[\mathcal{K}(A,m)]{} & \mathcal{K}(A,Y)
\end{array}$$

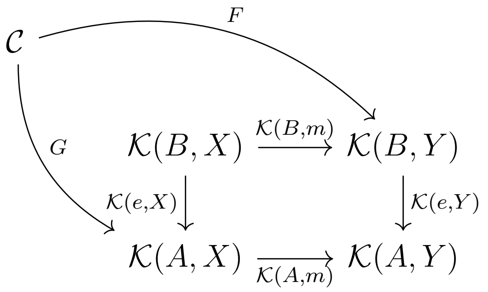
In ([10]), it was claimed that the converse in the following lemma only holds when the 2-category has cotensors with **2**. This argument indeed holds when one replaces chronic arrows by fully faithfull arrows, but using that each chronic is a monomorphism, the converse always holds:

Lemma 9. *Let $e \in \mathcal{K}(A,B)$ be acute and given a commutative diagram*

$$\begin{array}{ccc} A & \xrightarrow{e} & B \\ \Big\downarrow{\scriptstyle u} & & \Big\downarrow{\scriptstyle v} \\ X & \xrightarrow{m} & Y \end{array}$$

where m is chronic. Then there exists a unique $w : B \to X$ such that $u = w \circ e$ and $v = m \circ w$. Conversely, if a 1-cell satisfies this condition, it is acute.

Proof. The universal property of the pullback (on the objects) means precisely the first statement.
For the converse, consider a 1-category $\mathcal{C}$ and functors F, G such that the following diagram (in **Cat**) commutes:



We need to construct a functor $H : \mathcal{C} \to \mathcal{K}(B,X)$ such that F and G factor through it.
Let $C \in \mathcal{C}$, by the commutativity of the diagram and the hypothesis, there exists $h_C \in \mathcal{K}(B,X)$ such that

$$\begin{array}{ccc} A & \xrightarrow{e} & B \\ {\scriptstyle G(C)}\Big\downarrow & \swarrow{\scriptstyle h_C} & \Big\downarrow{\scriptstyle F(C)} \\ X & \xrightarrow[m]{} & Y \end{array}$$

commutes. Define $H(C) := h_C$. By definition of h_C is we have $H(C) \circ \mathcal{K}(B,m) = F(C)$, thus for $f \in \mathcal{C}(C,D)$, we have that $F(f)$ is a 2-cell from $F(C) = m \circ H(C)$ to $F(D) = m \circ H(D)$. Since m is chronic, $\mathcal{K}(B,m)$ is fully faithfull, thus there exists a unique 2-cell $h_f : H(C) \implies H(D)$ such that $F(f) = m \circ h_f$. So for each $f \in \mathcal{C}(C,D)$, define $H(f) = h_f$.
That H respects the identity morphism $Id_C \in \mathcal{C}(C,C)$ follows because $\mathcal{K}(B,m)(Id_{HC}) = Id_{FC}$ holds, but HC is the unique morphism which satisfies this equality by definition (or using that $\mathcal{K}(B,m)$ is fully faithfull). The *same* argument show that H respects composition, so H is indeed a functor.
That G factorizes through H follows from (again applying fully faithfullness)

$$m \circ G(f) = F(f) \circ e = m \circ H(f) \circ e.$$

And we conclude by saying that H is unique by construction, indeed: The 1-cell h_C is the unique morphism a such that

$$F(C) = \mathcal{K}(B,m) \circ a, G(C) = \mathcal{K}(e,X) \circ a,$$

by the hypothesis, which shows uniqueness of H on the objects. And the uniqueness on the morphisms follows by uniqueness of h_f such that $F(f) = m \circ h_f$.

So we have shown that the diagram is indeed a pullback which shows that e is acute. $\square$

Example 11. *In $\mathcal{K} = \mathbf{Cat}$, the acute arrows are the functors which are surjective on objects.*

Proof. Let $E : \mathcal{A} \to \mathcal{B}$ be a functor.
First assume that E is acute. We have that the following diagram is commutative:

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{E} & \mathcal{B} \\ \downarrow{\scriptstyle E} & & \downarrow{\scriptstyle Id} \\ Im(E) & \overset{M}{\hookrightarrow} & \mathcal{B} \end{array}$$

Since M is chronic (since by definition it makes $Im(E)$ into a full subcategory of $\mathcal{B}$), there exists a unique functor $G : \mathcal{B} \to Im(E)$ such that $E = G \circ E$ and $Id_{\mathcal{B}} = M \circ G$. Let $B \in \mathcal{B}$ be an object. So from

$$B = Id(B) = MG(B) = G(B),$$

we conclude that B is in the image of E (since $G(B)$ is). As this holds for all $B \in \mathcal{B}$, we have that E is indeed surjective on objects.
Now assume that E is surjective on objects and consider the following commutative diagram:

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{E} & \mathcal{B} \\ \downarrow{\scriptstyle F} & & \downarrow{\scriptstyle G} \\ \mathcal{X} & \overset{M}{\hookrightarrow} & \mathcal{Y} \end{array}$$

where M is chronic. For each $B \in \mathcal{B}$, choose an object $A^{(B)} \in \mathcal{A}$ such that $E(A^{(B)}) = B$. Now consider a morphism $f \in \mathcal{B}(B_1, B_2)$, this induces a morphism $G(f) \in \mathcal{Y}(G(B_1), G(B_2))$. Since

$$G(B) = GE(A^{(B)}) = MF(A^{(B)}),$$

we conclude from the (faith)fullness of M that there exists a (unique) morphism $g^{(f)} \in \mathcal{X}(F(A^{(B_1)}), F(A^{(B_2)}))$ such that $M(g^{(f)}) = f$. First notice that the following assignments define a functor H:

$$\begin{array}{rll} \mathcal{B} \xrightarrow{H} \mathcal{X} : & B & \mapsto A^{(B)} \\ & f & \mapsto g^{(f)} \end{array}$$

Indeed, that the identities and composition are preserved follows because G and M preserves the identities and composition and M moreover reflects the identities.
So we have a functor H which satisfies $M \circ H = G$ and $F = H \circ E$. So it remains to show the uniqueness of such H, but this follows by the uniqueness of $g^{(f)}$. $\square$

Lemma 10.

- *The composition of acute arrows is acute.*
- *Acute chronic arrows are isomorphisms.*
- *If $e \circ f$ is acute and f is either acute or an epimorphism, then is e acute.*

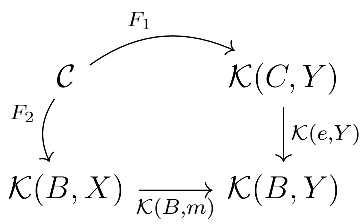
Proof. The composition is again clear since the composition of pullback squares is again a pullback square.
Assume $e : A \to B$ is both chronic and acute. Since $Id_B \circ e = e \circ Id_A$, there exists (by the previous lemma) an arrow $n : B \to A$ such that the following diagram commutes:

$$\begin{array}{ccc} A & \xrightarrow{e} & B \\ {\scriptstyle Id_A}\downarrow & \overset{n}{\swarrow} & \downarrow{\scriptstyle Id_B} \\ A & \xrightarrow{e} & B \end{array}$$

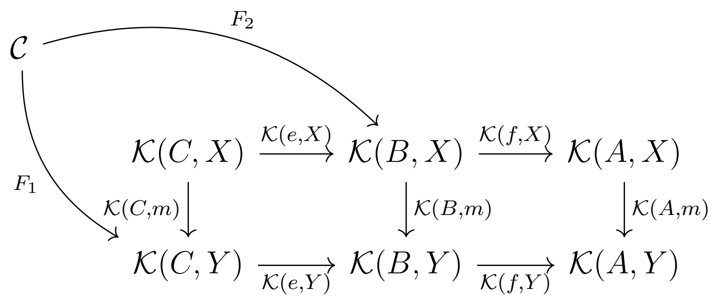
So this commutativity tells that e and n are inverses of each other.
Assume $e \circ f : A \to B \to C$ is acute and let $m : X \to Y$ be chronic. The acuteness of $e \circ f$ means that the following (outer) diagram is a pullback square:

$$\begin{array}{ccccc} \mathcal{K}(C,X) & \xrightarrow{\mathcal{K}(e,X)} & \mathcal{K}(B,X) & \xrightarrow{\mathcal{K}(f,X)} & \mathcal{K}(A,X) \\ {\scriptstyle \mathcal{K}(C,m)}\downarrow & & \downarrow{\scriptstyle \mathcal{K}(B,m)} & & \downarrow{\scriptstyle \mathcal{K}(A,m)} \\ \mathcal{K}(C,Y) & \xrightarrow[\mathcal{K}(e,Y)]{} & \mathcal{K}(B,Y) & \xrightarrow[\mathcal{K}(f,Y)]{} & \mathcal{K}(A,Y) \end{array}$$

So we have to show that when f is either an epimorphism of acute, that e is acute, i.e. the left square (of the above diagram) is a pullback square. If f is acute, the right square is a pullback square, so by some well-known pasting lemma, we can conclude that if also the outer square is a pullback square, the left one is a pullback square.
So it remains to show the case when f is an epi. So assume that the following diagram commutes:



So the following diagram commutes:



Since $e \circ f$ is acute, there exists a unique functor $F : \mathcal{C} \to \mathcal{K}(C,Y)$ such that

$$F_1 = \mathcal{K}(C,m) \circ F, \quad \mathcal{K}(f,X) \circ \mathcal{K}(e,X) \circ F = \mathcal{K}(f,X) \circ F_2.$$

So it remains to show that $\mathcal{K}(e,X) \circ F = F_2$:
For an object $c \in \mathcal{C}$, we have $F_2(c) \circ f = F(c) \circ e \circ f$. So we $F_2(c) = F(c) \circ e$ (since

f is an epimorphism).
For $\phi \in \mathcal{C}(c, d)$ we have

$$m \circ F(\phi) \circ e = F_1(\phi) \circ e = m \circ F_2(\phi).$$

By chronicness of m we then conclude $F(\phi) \circ e = F_2(\phi)$. So the claim is proven. $\square$

Lemma 11. *Let $\mathcal{K}$ be finitely complete. Let $e \in \mathcal{K}(A, B)$ be acute, then for all 2-cells $B \underset{\theta \Downarrow\Downarrow \phi}{\rightrightarrows} C$, if $\theta \circ e = \phi \circ e$, then $\theta = \phi$ and if $\theta \circ e$ is the identity 2-cell, then is θ the identity 2-cell.*

Proof. First assume $\theta \circ e = \phi \circ e$. Let $k : K \to B$ be the universal arrow which satisfies $\theta \circ k = \phi \circ k$ (informally this is the 2-dimensional analogue of the equalizer). So by universality of k, there exist some $u : A \to K$ such that $e = k \circ u$.
So because of the commutativity of

$$\begin{array}{ccc} A & \xrightarrow{e} & B \\ \big\downarrow u & & \big\downarrow Id_B \\ K & \xrightarrow{k} & B \end{array}$$

and using that e is acute and k chronic, there exist some $w : B \to K$ such that $Id_B = k \circ w$ and $u = w \circ e$. Since k is a monomorphism (by chronicness) and $Id_B = k \circ w$, we have that k is an isomorphism, thus $\theta = \phi$.
To show the second claim, the same strategy is used where we take k to be the universal arrow such that $\theta \circ k = Id$. $\square$

Chapter 2

Congruences

In this chapter we introduce the notion of a **congruence** in a 2-category. which is a generalization of an equivalence relation in a (1−)category. R. Street defined this notion of a congruence as a particular kind of internal functor (between internal categories). So therefore we first introduce internal category theory and later we show that the internal categories in **Cat** are precisely the **double categories** which gives us a explicit formulation of congruences in **Cat**.

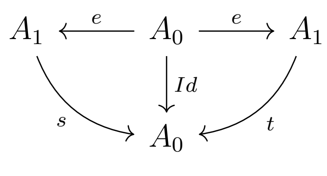
2.1 Internal category theory

Definition 12. *Let $\mathcal{C}$ be a category with pullbacks. An* ***internal category*** *$\mathcal{A}$ in $\mathcal{C}$ is a tuple (A_0, A_1, s, t, e, c) with*

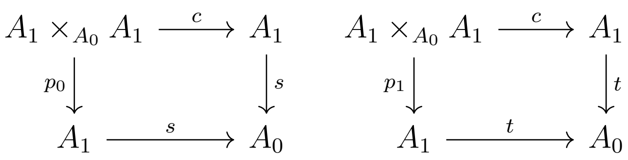
- *an object $A_0 \in \mathcal{C}$ called the* ***object of objects***
- *an object $A_1 \in \mathcal{C}$ called the* ***object of arrows***
- *morphisms $s, t : A_1 \to A_0$ called the* ***source*** *and* ***target***
- *a morphism $e : A_0 \to A_1$ called the* ***identity***
- *a morphism $c : A_1 \times_{A_0} A_1 \to A_1$ (where $A_1 \times_{A_0} A_1$ is the pullback of s and t) called the* ***composition***

which satisfies the usual axioms of a category, more precisely, the following diagrams commute:

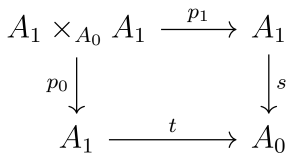
- *Axiom specifying the source and target of the identities:*



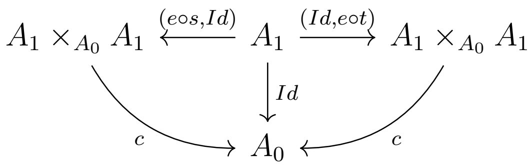
- *Axiom specifying the source and target of the composition:*



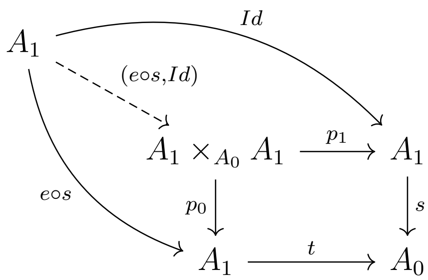
where p_0 and p_1 are such that the following diagram is a pullback square:

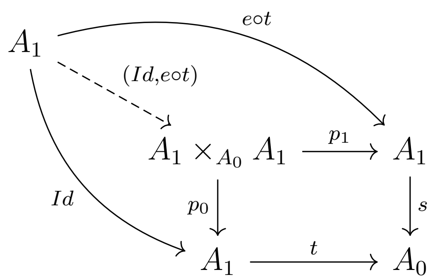


- *Unit axiom for composition:*

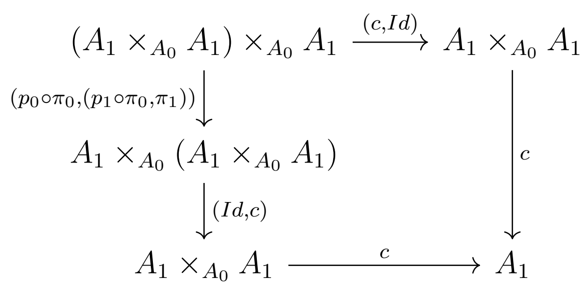


where $(e \circ s, Id)$ and $(Id, e \circ t)$ are the unique morphisms which complete the following diagrams:





- *Axiom of associativity:*



where the tuples $(Id, c), (c, Id)$ and $(p_0 \circ \pi_0, (p_1 \circ \pi_0, \pi_1))$ are the unique factorizations given by the universal property of the appropriate pullback (just as with the unit axiom of the composition) and π_0, π_1 are such that the following diagram is a pullback square:

$$\begin{array}{ccc} (A_1 \times_{A_0} A_1) \times_{A_0} A_1 & \xrightarrow{\pi_1} & A_1 \\ {\scriptstyle \pi_0}\downarrow & & \downarrow{\scriptstyle s} \\ A_1 \times_{A_0} A_1 & \xrightarrow{t \circ c} & A_0 \end{array}$$

Just as we could define a internal category in any category (with sufficient conditions), we can also define a *internal functor* between internal categories:

Definition 13. *Let $\mathcal{C}$ be a category with pullbacks. Let $\mathcal{A}$ and $\mathcal{B}$ be internal categories in $\mathcal{C}$. An* ***internal functor*** *$F : \mathcal{A} \to \mathcal{B}$ is a pair of morphisms $(F_0 : A_0 \to B_0, F_1 : A_1 \to B_1)$ which satisfies*

- $s \circ F_1 = F_0 \circ s, t \circ F_1 = F_0 \circ t$
- $F_1 \circ e = e \circ F_1$
- $F_1 \circ c = c \circ (F_1 \times_{F_0} F_1)$

Example 12. *The internal categories in* **Set** *correspond with the small categories and the internal functors correspond with the functors between those (small) categories.*

Proof. This is example is not needed in the sequel in the thesis, so for illustrating purposes we only show the correspondence with the internal categories. Let $\mathcal{C}$ be a small category. We define an internal category $\mathcal{A}^{\mathcal{C}} := (A_0, A_1, s, t, e, c)$ in **Set** as follows: A_0 is the set of objects in $\mathcal{C}$ (this is indeed a set since $\mathcal{C}$ is small). A_1 is the union of all hom-sets in $\mathcal{C}$, i.e. $A_1 := \bigcup_{x,y\in\mathcal{C}} \mathcal{C}(x, y)$ (this is also a set because each hom-set is a set and we take the union indexed by a set). The morphisms source s, target t and identity e morphisms are defined as follows:

$$s : A_1 \to A_0 : (f : x \to y) \mapsto x, \quad t : A_1 \to A_0 : (f : x \to y) \mapsto y, \quad e : A_0 \to A_1 : x \mapsto (Id_x \in \mathcal{C}(x, x)).$$

These are clearly functions. We now define the composition. The pullback of s along t is given by the (sub)set

$$A_1 \times_{A_0} A_1 = \{(f, g) \in A_1 \times A_1 | t \circ f = s \circ g\},$$

thus

$$c : A_1 \times_{A_0} A_1 \to A_1 : (f, g) \mapsto g \circ f,$$

is a well defined function. These functions clearly satisfy the axioms of an internal category and since all data consists of sets and functions, this indeed defines an internal category in **Set**.
Conversely consider an internal category $\mathcal{A} := (A_0, A_1, s, t, e, c)$ in **Set**. We define a small category $\mathcal{C}^{\mathcal{A}}$ as follows: Let the objects of $\mathcal{C}^{\mathcal{A}}$ be the elements in A_0 (so the objects form a set since A_0 is a set). Each hom-set $\mathcal{C}^{\mathcal{A}}(x, y)$ is given by

$$\mathcal{C}^{\mathcal{A}}(x, y) := \{f \in A_1 | s \circ f = x, t \circ f = y\}.$$

This is a set since A_1 is a set. The identity morphism Id_x correspond with $e(x)$. The composition of $f \in \mathcal{C}^{\mathcal{A}}(x, y)$ with $g \in \mathcal{C}^{\mathcal{A}}(y, z)$ is given by $g \circ f := c(f, g)$. The axioms of $\mathcal{A}$ being an internal category translates directly to $\mathcal{C}^{\mathcal{A}}$ being a (small) category.
It is clear that $(\mathcal{C}^{\mathcal{A}})^{\mathcal{C}} = \mathcal{C}$ and $(\mathcal{A}^{\mathcal{C}})^{\mathcal{A}} = \mathcal{A}$ which shows the claim. □

Remark 3. *In a straightforward way, one can define the notion of internal natural transformations. Just as* **Cat** *is a 2-category, analogously we can conclude that for a fixed category (with sufficient conditions), its internal categories together with the internal functors and internal natural transformations form a 2-category.*

2.1.1 Double categories

Definition 14. *A double category $\mathcal{D}$ consists of:*

- *a class of* 0*-cells* $\mathcal{D}_0$,
- *for each $A, B \in \mathcal{D}_0$, a class of horizontal* 1*-cells $Hor(\mathcal{D})(A, B)$ such that for each $A \in \mathcal{D}_0$, there exists a identity* 1*-cell $Id^H_A \in Hor(\mathcal{D})(A, A)$ and a composition function*

$$\circ^H : Hor(\mathcal{D})(A, B) \times Hor(\mathcal{D})(B, C) \to Hor(\mathcal{D})(A, C).$$

- *for each $A, B \in \mathcal{D}_0$, a class of vertical* 1*-cells $Ver(\mathcal{D})(A, B)$ such that for each $A \in \mathcal{D}_0$, there exists a identity* 1*-cell $Id^V_A \in Ver(\mathcal{D})(A, A)$ and a composition function*

$$\circ^V : Ver(\mathcal{D})(A, B) \times Ver(\mathcal{D})(B, C) \to Ver(\mathcal{D})(A, C).$$

- *if $f : A \to B, g : C \to D$ are horizontal* 1*-cells and $u : A \to C, v : B \to D$ are vertical* 1*-cells, a class $Dbl(\mathcal{D})(f, g, u, v)$ of* 2*-cells. We denote a* 2*-cell $\alpha \in Dbl(\mathcal{D})(f, g, u, v)$ by a square:*

$$\begin{array}{ccc} A & \overset{f}{\to} & B \\ {}^{u}\downarrow & \alpha & \downarrow^{v} \\ C & \underset{g}{\to} & D \end{array}$$

such that if $h : B \to E$ and $i : D \to F$ are horizontal 1*-cells and $w : E \to F$ a vertical* 1*-cell, there is a horizontal composition function*

$$\bullet^H : Dbl(\mathcal{D})(f, g, u, v) \times Dbl(\mathcal{D})(h, i, v, w) \to Dbl(\mathcal{D})(h \circ^H f, i \circ^H g, u, w).$$

This composition is denoted by pasting the squares horizontally, i.e.

$$\begin{array}{ccccc} A & \overset{f}{\to} & B & \overset{h}{\to} & E \\ {}^{u}\downarrow & \alpha & \downarrow^{v} & \beta & \downarrow^{w} \\ C & \underset{g}{\to} & D & \underset{i}{\to} & F \end{array}$$

And there exists also a vertical composition of 2*-cells*

$$\bullet^V : Dbl(\mathcal{D})(f, g, u, v) \times Dbl(\mathcal{D})(g, j, x, y) \to Dbl(\mathcal{D})(f, j, x \circ^V u, y \circ^V w),$$

which is visualized as:

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {}^{u}\downarrow & \alpha & \downarrow^{v} \\ C & \xrightarrow[g]{} & D \\ {}^{x}\downarrow & \beta & \downarrow^{y} \\ E & \xrightarrow[j]{} & F \end{array}$$

These data must satisfy the following axioms:

- $\mathcal{D}_0$ *together with the horizontal* 1*-cells is a category, i.e. the horizontal composition* $\circ^H$ *is associative and the composition of a horizontal* 1*-cell* $f : A \to B$ *with horizontal identity* 1*-cells (on both sides) is again the* 1*-cell, i.e.* $Id_B^H \circ^H f = f = f \circ^H Id_A$.

- $\mathcal{D}_0$ *together with the vertical* 1*-cells is a category, i.e. the vertical composition* $\circ^V$ *is associative and the composition of a vertical* 1*-cell* $g : A \to C$ *with vertical identity* 1*-cells (on both sides) is again the* 1*-cell, i.e.* $Id_C^V \circ^V g = g = g \circ^V Id_A$.

- *The vertical and horizontal composition are both associative and both composition with the identity* 2*-cells preserves the original* 2*-cell.*

- *The interchange law holds, that is: The following diagram is well-defined:*

$$\begin{array}{ccccc} A & \xrightarrow{f} & B & \xrightarrow{h} & E \\ {}^{u}\downarrow & \alpha & \downarrow^{v} & \beta & \downarrow^{w} \\ C & \xrightarrow[g]{} & D & \xrightarrow[i]{} & F \\ {}^{x}\downarrow & \gamma & \downarrow^{y} & \delta & \downarrow^{z} \\ G & \xrightarrow[j]{} & H & \xrightarrow[k]{} & I \end{array}$$

Example 13. *Every* 2*-category* $\mathcal{K}$ *can be considered as a double category* $\mathcal{D}$ *by having no non-trivial vertical* 2*-cells, more precisely:*

- $\mathcal{D}_{\prime} := \mathcal{K}_0$.

- $Hor(\mathcal{D})(A, B) := \mathcal{K}(A, B)_0$.

- $$Ver(\mathcal{D})(A, A) := \begin{cases} \{Id_A\} & \text{if } A = B, \\ \emptyset & \text{else.} \end{cases}$$

- $Dbl(\mathcal{D})(f, g, Id, Id) := \mathcal{K}(A, B)(f, g)$.

Example 14. *The internal categories in* **Cat** *correspond with the small double categories.*

Proof. A internal category $\mathcal{A} := (A_0, A_1, s, t, e, c)$ in **Cat** consists in particular of (small) categories A_0 and A_1. Let $(A_0)_0$ (resp. $(A_1)_0$) be the set of objects of A_0 (resp. A_1) and $(A_0)_1$ (resp. $(A_1)_1$) be the set of all morphisms in A_0 (resp. A_1). We define a double category $\mathcal{D}^{\mathcal{A}}$ as follows:

- $\mathcal{D}^{\mathcal{A}}{}_0 := (A_0)_0$,
- $Ver(\mathcal{D}^{\mathcal{A}}) := (A_0)_1$,
- $Hor(\mathcal{D}^{\mathcal{A}}) := (A_1)_0$,
- $Dbl(\mathcal{D}^{\mathcal{A}}) := (A_1)_1$.

The composition of vertical 1-cells is the composition in A_0 (under the assumption that they are compatible). The composition of vertical 1-cells is defined using $c : A_1 \times_{A_0} A_1 \to A_1$ as follows: Let $f, g \in Hor(\mathcal{D}) = (A_1)_0$ such that $t(f) = s(g)$ (notice that this is well-defined because $s, t : A_1 \to A_0$ are functors and f, g objects in A_1). Then we define $g \circ^V f := c(f, g)$. Since both the composition in A_0 and c are associative and respect the identities, so do the horizontal and vertical composition. We now define the vertical and horizontal composition of double cells (which is analogous to that of the 1-cells): The vertical composition of double cells is defined as the composition in A_1 (since the double cells are the morphisms in A_1) and the horizontal composition is defined using c. Again, since the composition in A_1 and c satisfy the associativity and identity axioms, we have that these compositions satisfy the correct axioms. It remains to check the interchange law, but this follows from the functoriality of the composition functor.
The converse is analogous by reversing the definitions. For example, if $\mathcal{D}$ be a double category. The objects in the corresponding internal category $\mathcal{A}^{\mathcal{D}} := (A_0, A_1, s, t, e, c)$ are the following:

- A_0 is the category whose objects are the elements of $\mathcal{D}$ (i.e. $(A_0)_0 := \mathcal{D}_0$) and whose morphisms are the vertical arrows (i.e. $(A_0)_1 := Vert(\mathcal{D})$).
- A_1 is the category whose objects are the horizontal arrows (i.e. $(A_1)_0 := Hor(\mathcal{D})$) and whose morphisms are the double cells (i.e. $(A_1)_1 := Dbl(\mathcal{D})$).

□

2.2 Congruences

Before defining a congruence, we introduce the following definition:

Definition 15. *A span $\mathcal{A} \xleftarrow{F} \mathcal{B} \xrightarrow{G} \mathcal{C}$ (in* **Cat***) is a* ***discrete fibration*** *if*

- *("Unique F-left") For each $f \in \mathcal{A}(FB, A)$, there exists a unique morphism $f^F \in \mathcal{B}(B, A^F)$ such that $F(f^F) = f, G(f^F) = Id$. So in particular, $F(A^F) = A$ and $G(A^F) = G(B)$.*
- *("Unique G-left") For each $g \in \mathcal{C}(C, GB)$, there exists a unique morphism $g^G \in \mathcal{B}(C^G, B)$ such that $G(g^G) = g, F(g^G) = Id$. So in particular, $G(C^G) = C$ and $F(C^G) = F(B)$.*
- *("Bimodule condition") For each morphism $f \in \mathcal{B}(B_1, B_2)$, we have $B_2^F = B_1^G$ and $B_1 \xrightarrow{h^F} (B_2^F = B_1^G) \xrightarrow{h^G} B_2 = B_1 \xrightarrow{h} B_2$.*

*A span $A \xleftarrow{f} B \xrightarrow{g} C$ in a 2-category $\mathcal{K}$ is a **discrete fibration** if it representably so, i.e. for each $K \in \mathcal{K}$*

$$\mathcal{K}(K,A) \xleftarrow{\mathcal{K}(K,f)} \mathcal{K}(K,B) \xrightarrow{\mathcal{K}(K,g)} \mathcal{K}(K,C),$$

*is a discrete fibration (in **Cat**).*

Definition 16. *A **congruence** $\mathbb{E}$ on an object $A \in \mathcal{K}$ is an internal functor $j : E \to F$ (in $\mathcal{K}_0$) such that*

- *$E_0 = A = F_0$, $j_0 = Id_A$*
- *$j_1 : E_1 \to F_1$ is chronic*
- *F is an equivalence relation on A, i.e. $(s,t) : F_1 \to A \times A$ is a monomorphism and there exists a morphism $p \in \mathcal{K}(F_1, F_1)$ such that $s \circ p = t, t \circ p = s$.*
- *The span (s_F, F_1, t_F) is a discrete fibration*

We denote by $Cong(\mathcal{K})$ the 2-category with as its 0-cells the congruences, a 1-cell from $(j : E^j \to F^j)$ to $(k : E^k \to F^k)$ is given by 1-cells e, f in $\mathcal{K}$ such that the following diagram commutes:

$$\begin{array}{ccc} E_1^j & \xrightarrow{j_1} & F_1^j \\ \big\downarrow{\scriptstyle e} & & \big\downarrow{\scriptstyle f} \\ E_1^k & \xrightarrow{k_1} & F_1^k \end{array}$$

and a 2-cell from (e,f) to $(\tilde{e}, \tilde{f})$ is given by 2-cells $\alpha : e \implies \tilde{e}$ and $\beta : f \implies \tilde{f}$ such that the following diagram commutes:

$$\begin{array}{ccc} E_1^j & \xrightarrow{j_1} & F_1^j \\ e \big(\Rightarrow \alpha \big) \tilde{e} & & f \big(\Rightarrow \beta \big) \tilde{f} \\ E_1^k & \xrightarrow{k_1} & F_1^k \end{array}$$

We are now going to associate to each 1-cell a congruence. Let $\mathcal{K}$ be finitely complete and $f \in \mathcal{K}(A,B)$. Consider the following diagrams:

$$\begin{array}{ccc} E_1 & \xrightarrow{s^E} & A \\ \big\downarrow{\scriptstyle t^E} & & \big\downarrow{\scriptstyle f} \\ A & \xrightarrow{f} & B \end{array} \qquad \begin{array}{ccc} F_1 & \xrightarrow{s^F} & A \\ \big\downarrow{\scriptstyle t^F} & & \big\downarrow{\scriptstyle f} \\ A & \xrightarrow{f} & B \end{array}$$

Where the left (resp. right) square has the pullback (resp. comma) property. Also notice since $f = f$, we get unique morphisms $e^E : A \to E_1$ and $e^F : A \to F_1$ such that

$$s^E \circ e^E = Id_A = t^E \circ e^E, s^F \circ e^F = Id_A = t^F \circ e^F. \tag{2.1}$$

For the composition, we need a morphism $c^E : E_1 \times_A E_1 \to E_1$ (and analogously for c^F) where $E_1 \times_A E_1$ is the pullback of s^E along t^E. Denote by p^s_E, p^t_E the projections, thus $s \circ p^s_E = t \circ p^t_E$. So by the universal property E_1 (as the pullback of f along f with projections s, t), there exists a unique morphism $c^E : E_1 \times_A E_1 \to E_1$ such that

$$s^E \circ c^E = s^E \circ p^s_E, \quad t^E \circ c^E = t^E \circ p^t_E. \tag{2.2}$$

This data gives us internal categories $E \equiv (A, E_0, s^E, t^E, e^E, c^E)$ and $F \equiv (A, F_0, s^F, t^F, e^F, c^F)$ indeed: equation (2.1) gives us the axiom specifying the source and target of the identity morphisms, equation (2.2) gives us the axiom that composition respects the source and target. So it remains to show:

- Composition respects source and target: $c^E \circ (Id_{E_1}, e \circ s) = Id_{E_1} = c^E \circ \left(e^E \circ t^E, Id_{E_1}\right)$.
- Associativity of composition: $c^E \circ \left(Id_{E_1}, c^E\right) = c^E \circ (c^E, Id_{E_1})$.

The first equation holds because Id_{E_1} is the unique morphism such that $s^E = s^E \circ Id_{E_1}, t^E = t^E \circ Id_{E_1}$, but $c^E \circ (Id_{E_1}, e \circ s)$ also satisfies these equalities, indeed:

$$s^E = s^E \circ Id_{E_1} = s^E \circ p^s_E \circ (Id_{E_1}, e \circ s) = s^E \circ c^E \circ (Id_{E_1}, e \circ s),$$
$$t^E = t^E \circ Id_{E_1} = t^E \circ p^t_E \circ (Id_{E_1}, e \circ s) = s^E \circ c^E \circ (Id_{E_1}, e \circ s),$$

where the thirth equality holds by definition of $(Id_{E_1}, e \circ s)$ and the fourth holds by definition of c^E. The associativity of the composition is analogous.
Using the comma property of F_1, we get a 1-cell $j_1 : E_1 \to F_1$. This is moreover chronic by the pullback property of F_1. Together with Id_A, this becomes an internal functor $j = (Id_A, j_1) : E \to F$.
That E_1 (resp. F_1) is an equivalence relation on A is immediate as it is a pullback over f with itself which has as its domain A. More precisely: $(s^E, t^E) : E \to A \times A$ is mono by the unique factorisation and since $f \circ s = f \circ t$, there exists a (unique) $p \in \mathcal{K}(E, E)$ for which $t \circ p = s$ and $s \circ p = t$.

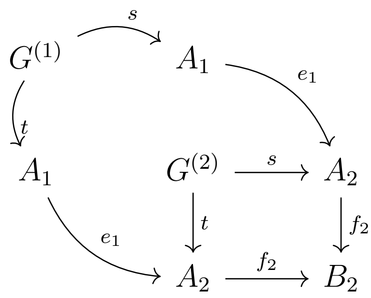
Proposition 3. *The congruence associated to a morphism induces a 2-functor*

$$\mathbf{E} : Fun_2(2, \mathcal{K}) \to Cng(\mathcal{K}).$$

Proof. A 1-cell between 0-cells $f_1 : A_1 \to B_1$ and $f_2 : A_2 \to B_2$ (in $[2, \mathcal{K}]$) is given by a commutative diagram

$$\begin{array}{ccc} A_1 & \xrightarrow{f_1} & B_1 \\ \downarrow{\scriptstyle e_1} & & \downarrow{\scriptstyle e_2} \\ A_2 & \xrightarrow{f_2} & B_2 \end{array}$$

We denote by $j_1^{(i)} : E_1^{(i)} \to F_1^{(i)}$ the data of the congruence $E(f_i)$ on A_i $(i = 1, 2)$. Denote by $G^{(i)}$ either $F_1^{(i)}$ or $E_1^{(i)}$. Since $e_2 \circ f_1 = f_2 \circ e_1$ and $f_1 \circ t = f_1 \circ s$, we have that the following diagram is commutative:



So by the universal property of $G^{(2)}$, there exists unique $\phi^{(G)} : G^{(1)} \to G^{(2)}$ such that

$$t^{(2)} \circ \phi = e_1 \circ t^{(1)}, \quad s^{(2)} \circ \phi = e_1 \circ s^{(1)}.$$

So it remains to show that these ϕ's define a map of congruences, i.e. the following diagram is commutative:

$$\begin{array}{ccc} E_1^{(1)} & \xrightarrow{j_1^{(1)}} & F_1^{(1)} \\ \downarrow{\scriptstyle \phi^{(E)}} & & \downarrow{\scriptstyle \phi^{(F)}} \\ E_1^{(2)} & \xrightarrow{j_1^{(2)}} & F_1^{(2)} \end{array}$$

Since $(s^{F,(2)}, t^{F,(2)}) : F_1^{(2)} \to A_2 \times A_2$ is a monomorphism, it suffices to show

$$\begin{cases} s^{F,(2)} \circ \phi^{(F)} \circ j_1^{(1)} = s^{F,(2)} \circ j_1^{(2)} \circ \phi^{(E)} \\ t^{F,(2)} \circ \phi^{(F)} \circ j_1^{(1)} = t^{F,(2)} \circ j_1^{(2)} \circ \phi^{(E)} \end{cases}$$

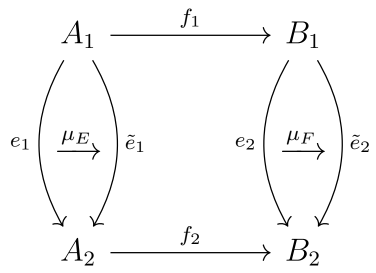
That these equalities hold follow from:

$$\begin{array}{rcl} s^{F,(2)} \circ \phi^{(F)} \circ j_1^{(1)} & = & e_1 \circ s^{F,(1)} \circ j_1^{(1)}, \quad \text{by definition } \phi^{(F)} \\ & = & e_1 \circ s^{E,(1)}, \qquad \text{by definition } j_1^{(1)} \\ & = & s^{E,(2)} \circ \phi^{(E)}, \qquad \text{by definition } \phi^{(E)} \\ & = & s^{F,(2)} \circ j_1^{(2)} \circ \phi^{(E)}, \quad \text{by definition } j_1^{(2)} \end{array}$$

The exactly same argument shows the second equality (so s replaced by t).
A 2-cell μ in $[2, \mathcal{K}]$

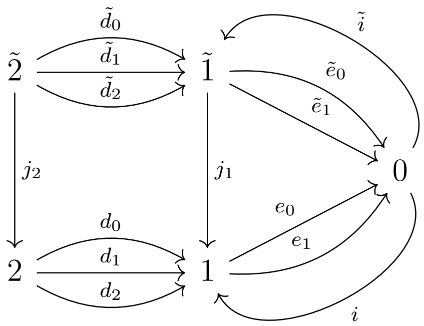
$$\begin{array}{ccc} A_1 & \xrightarrow{f_1} & B_1 \\ \downarrow{\scriptstyle e_1} & & \downarrow{\scriptstyle e_2} \\ A_2 & \xrightarrow{f_2} & B_2 \end{array} \quad \xrightarrow{\mu} \quad \begin{array}{ccc} A_1 & \xrightarrow{f_1} & B_1 \\ \downarrow{\scriptstyle \tilde{e}_1} & & \downarrow{\scriptstyle \tilde{e}_2} \\ A_2 & \xrightarrow{f_2} & B_2 \end{array}$$

is given by a commutative diagram:



In an analogous way, we define it on 2-cells and the functoriality is then a standard uniqueness argument. □

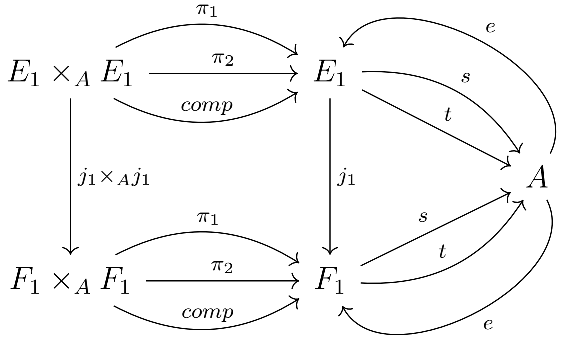
We will now introduce the notion of a quotient for a congruence. Let $\mathcal{D}$ be the (discrete 2-)category generated by:



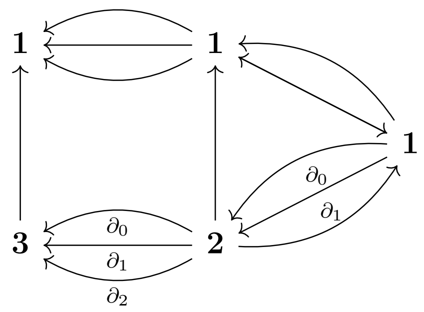
subject to the *simplicial identities*, more precisely:

$$e_0 \circ i = Id_0, e_1 \circ i = Id_0, e_0 \circ d_0 = e_0 \circ d_2, e_1 \circ d_1 = e_1 \circ d_2, e_0 \circ d_1 = e_1 d_0$$
$$\tilde{e}_0 \circ \tilde{i} = Id_0, \tilde{e}_1 \circ \tilde{i} = Id_0, \tilde{e}_0 \circ \tilde{d}_0 = \tilde{e}_0 \circ \tilde{d}_2, \tilde{e}_1 \circ \tilde{d}_1 = \tilde{e}_1 \circ \tilde{d}_2, \tilde{e}_0 \circ \tilde{d}_1 = \tilde{e}_1 \tilde{d}_0$$

A congruence $j : E \to F$ on A becomes a (2-)functor $E : \mathcal{D} \to \mathcal{K}$ by assigning the diagram of $\mathcal{D}$ to



Let $J : \mathcal{D}^{op} \to \mathbf{Cat}$ be the functor which assigns the above diagram to



where $\mathbf{1}, \mathbf{2}$ and $\mathbf{3}$ are the *free arrow categories* with $1, 2$ and resp. 3 objects which we denote by $\star_1, \star_2, \star_3$. The maps δ_i are given by:

$$\begin{aligned}
\partial_0 : \mathbf{1} \to \mathbf{2} \quad &: \quad \star_1 \mapsto \star_1 \\
\partial_1 : \mathbf{1} \to \mathbf{2} \quad &: \quad \star_1 \mapsto \star_2 \\
\partial_0 : \mathbf{2} \to \mathbf{3} \quad &: \quad \star_i \mapsto \star_i \quad (i = 1, 2) \\
\partial_1 : \mathbf{2} \to \mathbf{3} \quad &: \quad \star_i \mapsto \star_{i+1} \quad (i = 1, 2) \\
\partial_2 : \mathbf{2} \to \mathbf{3} \quad &: \quad \star_1 \mapsto \star_3; \quad \star_2 \mapsto \star_1
\end{aligned}$$

Definition 17. *A* ***quotient for a congruence*** *E on A is the J-indexed colimit $col(J, E)$ for $E : \mathcal{D} \to \mathcal{K}$.*

A quotient for a congruence E on A can be characterized as 0-cell $Q \in \mathcal{K}$ together with a 1-cell $q : A \to Q$ and a 2-cell $\tau : qs \to qt$ which are universal among those (Q, q, τ) satisfying some universality properties. We now spell out its 1-dimensional property:
Since $col(J, E)$ is the object defined as the object for which for every $X \in \mathcal{K}$, there is an equivalence of categories

$$\mathcal{K}(col(J, E), X) \cong Nat_2(J, \mathcal{K}(E(-), X)),$$

natural in X.
Let $X \in \mathcal{K}$ be an object and $\alpha : J \to \mathcal{K}(E(-), X)$ be a natural transformation. Since $J(0)$ is the terminal category and $E(0) = A$, α_0 correspond to a 1-cell $g : A \to X$. Since the map $\alpha_{\tilde{1}}$ has domain $J(\tilde{1}) = E_1$ and codomain $J(\tilde{1}) = \mathbf{1}$, $\alpha_{\tilde{1}}$ corresponds with a 1-cell $E_1 \to X$. By the (1-dimensional) naturality of α, we have that the following diagram commutes:

$$\begin{array}{ccc}
\mathbf{1} = J(\tilde{0}) & \xrightarrow{\alpha_{\tilde{0}}} & \mathcal{K}(E(0), X) = \mathcal{K}(A, X) \\
\big\downarrow{\scriptstyle J(\tilde{d}_0)} & & \big\downarrow{\scriptstyle \mathcal{K}(E(d_0), 0) = - \circ s} \\
\mathbf{1} = J(\tilde{1}) & \xrightarrow{\alpha_{\tilde{1}}} & \mathcal{K}(E(\tilde{1}), X) = \mathcal{K}(E_1, X)
\end{array}$$

This means precisely that $\alpha_{\tilde{1}}$ is given by $g \circ s$ (or equally $g \circ t$).
Since the map α_1 has domain $J(1) = \mathbf{2}$ and codomain $\mathcal{K}(E(1), X)$ (with $E(1) = F_1$), it corresponds with a natural transformation γ between arrows $F_1 \to X$. By the (1-dimensional) naturality of α, we have that the following diagram commutes:

$$\begin{array}{ccc}
\mathbf{2} = J(1) & \xrightarrow{\alpha_1} & \mathcal{K}(E(1), X) = \mathcal{K}(F_1, X) \\
\big\downarrow{\scriptstyle J(j_1)} & & \big\downarrow{\scriptstyle \mathcal{K}(E(j_1), 0) = - \circ j_1} \\
\mathbf{1} = J(\tilde{1}) & \xrightarrow{\alpha_{\tilde{1}}} & \mathcal{K}(E(\tilde{1}), X) = \mathcal{K}(E_1, X)
\end{array}$$

This means precisely that the identity 1-cell from $g \circ t^E$ to $g \circ s^E$ equals $\gamma \circ j_1$.

Proposition 4. *Let $\mathcal{K}$ be finitely cocomplete, then every congruence has a quotient.*

Proof. This is clear since the quotient of a congruence is a finite colimit. □

Definition 18. *A 1-cell $q : A \to Q$ is a* ***quotient map*** *when there exists a congruence E on A and a 2-cell $\tau : qs \to qt$ such that (Q, q, τ) form a quotient for E.*

Some usefull properties are:

Proposition 5. *Let $\mathcal{K}$ be finitely complete. An arrow q is a quotient map if and only if Q, q, λ provide a quotient for the congruence $\mathbf{E}(q)$.*

Proposition 6. *If a congruence is a congruence associated with some arrow, then it is the congruence associated with its quotient map (provided this quotient map exists).*

Proposition 7. *Every quotient map is acute.*

2.3 Regular and exact categories

Definition 19. *A 2-category $\mathcal{K}$ is* ***regular*** *if*

- *all finite 2-limits exists,*
- *each 1-cell f factors as $f = me$ with m chronic and e acute,*
- *the pullback of an acute arrow is acute.*

Example 15. Cat *is regular.*

Proof. We already know that **Cat** has all (finite) 2-limits. Since the chronics in **Cat** are the functors which are injective on objects and fully faithfull and the acutes are the functors which are essentially surjective on objects, this is just restating that those form a factorization system on **Cat**. So it remains to show that the pullback of an acute is acute:
Let $E : A \to B$ be essentially surjective on objects and let $F : C \to B$ be a functor. So the pullback of E along F is given by (as a category/object)

$$A \times_B C = \{(a, c) \in A \times C | E(a) = F(c)\},$$

and with the functors $A \times_B C \to A, A \times_B C \to C$ the projections. We have to show that the projection on C is essentially surjective so let $x \in C$. Since E is essentially surjective there exists some $a \in A$ such that $E(a) \cong F(x)$, thus (a, x) lies in the pullback and its projection on C is given by x which shows the claim. □

Lemma 12. *In a regular 2-category, the product of acute arrows is acute.*

Proof. Let $A_i \xrightarrow{e_i} B_i$, with $i = 1, 2$, be acute. For each $X, Y \in \mathcal{K}$, the following diagrams are (clearly) pullback squares:

$$\begin{array}{ccc} A_i \times X & \xrightarrow{e_i \times Id_X} & B_i \times X \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ A_i & \xrightarrow{e_i} & B_i \end{array} \qquad \begin{array}{ccc} Y \times A_i & \xrightarrow{Id_Y \times e_i} & Y \times B_i \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ A_i & \xrightarrow{e_i} & B_i \end{array}$$

Since e_i acute, $e_i \times Id_{A_1}$ and $Id_{A_2} \times e_i$ are acute (as they are the pullback of e_i), thus (since the composition of acute arrows is acute), we have that $e_1 \times e_2 = (e_1 \times Id_{A_2}) \circ (Id_{A_1} \times e_2)$ is acute. □

Definition 20. *A* 2*-category* $\mathcal{K}$ *is exact when it is regular and each congruence is the congruence associated with some arrow.*

Ross Street originally claimed that in exact 2-categories, every congruence has a quotient, but in ([4]), it was shown that there was a flaw in proof because not every acute arrow (in a regular category) is a quotient map.

Example 16. Cat *is an exact* 2*-category.*

Proof. We already know that **Cat** is regular. So let $\mathbb{E} \equiv (j : E \to F)$ be a congruence on $A \in$ **Cat**. We have shown that the internal categories and internal functors in **Cat** are the double functors and categories. So $E_0 = A = F_0$. By chronicness of $j_1 : E_1 \to F_1$, we can consider E_1 as a full subcategory of F_1. We call the objects of E_1 *trite vertical arrows* (notice here we call it arrows because E_1 is considered as the arrows in E). Objects $a_1, a_2 \in A$ are *equivalent* if there exists a trite vertical arrow between them. This is indeed an equivalence relation since E_1 is an equivalence relation from which we have that there exists at most one trite vertical arrow between a_1 and a_2 and it automatically is an isomorphism. This equivalence extends to the vertical arrows which defines a category Q and a functor $q : A \to Q$. It then follows that q is a quotient for $\mathbb{E}$. □

Example 17. *For a* 2*-category* $\mathcal{C}$*, the* 2*-functor* 2*-category* $Fun_2(\mathcal{C}^{op}, \mathbf{Cat})$ *is exact.*

Chapter 3

Two-dimensional sheaf theory

Definition 21. *A **topology** $\mathcal{T}$ on a 2-category $\mathcal{C}$ is a function which assigns to each object $U \in \mathcal{C}$ a collection $\mathcal{T}(U)$ consisting of chronic subfunctors*[1] *of $\mathcal{C}(-,U)$ such that:*

- *For each $U \in \mathcal{C}$, $\mathcal{C}(-,U) \in \mathcal{T}(C)$.*
- *If $R \in \mathcal{T}(U)$ and $f \in \mathcal{C}(V,U)$, then is the pullback (in $Fun_1[\mathcal{C}^{op}, \mathbf{Cat}]$) of R along $\mathcal{C}(-,f)$ in $\mathcal{T}(V)$, i.e. consider the following pullback square:*

$$\begin{array}{ccc} R_f & \longrightarrow & R \\ \big\downarrow & & \big\downarrow \\ \mathcal{C}(-,V) & \xrightarrow{\mathcal{C}(-,f)} & \mathcal{C}(-,U) \end{array}$$

  Then we have R_f in $\mathcal{T}(V)$.

- *Let R be a chronic subfunctor of $\mathcal{C}(-,U)$ and let $S \in \mathcal{T}(U)$. If for every $D \in \mathcal{C}$ and $f \in S(D)$, we have that $R_f \in \mathcal{T}(D)$, then is $R \in \mathcal{T}(U)$.*

*An element in $\mathcal{T}(U)$ is called a **covering crible** of U, a **covering sieve** of U or simply a U**-crible**.*
*A 2-category $\mathcal{C}$ with a topology $\mathcal{T}$ is called a 2**-site**.*

In order to keep the notation shorter, we write $[\mathcal{A}, \mathcal{B}]$ for the 2-category $Fun_2(\mathcal{A}, \mathcal{B})$ of 2-functors.

Definition 22. *A 2-**sheaf** on a 2-site $(\mathcal{C}, \mathcal{T})$ is a 2-functor $F : \mathcal{C}^{op} \to \mathbf{Cat}$ such that each covering sieve $R \to \mathcal{C}(-,U)$ induces an isomorphism*

$$[\mathcal{C}^{op}, \mathbf{Cat}]\,(R, F) \cong [\mathcal{C}^{op}, \mathbf{Cat}]\,(\mathcal{C}(-,U), F).$$

*If $\mathcal{K}$ is a 2-category, a $\mathcal{K}$**-valued** 2**-sheaf** is a 2-functor $F : \mathcal{C}^{op} \to \mathcal{K}$ such that for all $X \in \mathcal{K}$, the 2-functor*

$$\mathcal{K}(X,-) \circ F : \mathcal{C}^{op} \to \mathbf{Cat},$$

is a 2-sheaf for $\mathcal{T}$.

[1] By a chronic subfunctor of $\mathcal{C}(-,U)$ we mean a functor $R : \mathcal{C}^{op} \to \mathbf{Cat}$ together with a natural transformation $R \to \mathcal{C}(-,U)$ which is chronic in $Fun_2(\mathcal{C}^{op}, \mathbf{Cat})$

Lemma 13. *A* **Cat***-valued* 2*-sheaf is just a* 2*-sheaf.*

Proof. Let F be a **Cat**-valued 2-sheaf, so for each X we have that $\mathbf{Cat}(X,-)\circ F$ is a 2-sheaf, thus for $X=\mathbf{1}$, we have $F=\mathbf{Cat}(\mathbf{1},-)$ is a 2-sheaf.
For the converse, first notice that by cartesian closedness of **Cat** we have:

$$[\mathcal{C}^{op},\mathbf{Cat}](F,[\mathcal{C}^{op},\mathbf{Cat}](G,H))\cong[\mathcal{C}^{op},\mathbf{Cat}](G,[\mathcal{C}^{op},\mathbf{Cat}](F,H)),$$

indeed, this holds in **Cat** (and thus in $[\mathcal{C}^{op},\mathbf{Cat}]$) because

$$\mathbf{Cat}(A,\mathbf{Cat}(B,C))\cong\mathbf{Cat}(A\times B,C)\cong\mathbf{Cat}(B\times A,C)\cong\mathbf{Cat}(B,\mathbf{Cat}(A,C)).$$

Denote by $\underline{X}_{const}$ the constant functor $\mathcal{C}^{op}\to\mathbf{Cat}:C\mapsto X$, we then have a natural isomorphism $[\mathcal{C}^{op},\mathbf{Cat}](\underline{X}_{const},F)\cong F\circ\mathbf{Cat}(X,-)$:
So it remains to show the following:

$$[\mathcal{C}^{op},\mathbf{Cat}](R,\mathbf{Cat}(X,F(-)))\cong[\mathcal{C}^{op},\mathbf{Cat}](\mathcal{C}(-,U),\mathbf{Cat}(X,F(-))),$$

for each covering crible $R\to\mathcal{C}(-,U)$. This follows from the following computation:

$$\begin{aligned}
[\mathcal{C}^{op},\mathbf{Cat}](R,\mathbf{Cat}(X,F(-))) &\cong [\mathcal{C}^{op},\mathbf{Cat}](R,[\mathcal{C}^{op},\mathbf{Cat}](\underline{X}_{const},F))\\
&\cong [\mathcal{C}^{op},\mathbf{Cat}](\underline{X}_{const},[\mathcal{C}^{op},\mathbf{Cat}](R,F))\\
&\cong [\mathcal{C}^{op},\mathbf{Cat}](\underline{X}_{const},[\mathcal{C}^{op},\mathbf{Cat}](\mathcal{C}(-,U),F))\\
&\cong [\mathcal{C}^{op},\mathbf{Cat}](\mathcal{C}(-,U),[\mathcal{C}^{op},\mathbf{Cat}](\underline{X}_{const},F))\\
&\cong [\mathcal{C}^{op},\mathbf{Cat}](\mathcal{C}(-,U),\mathbf{Cat}(X,F(-)))
\end{aligned}$$

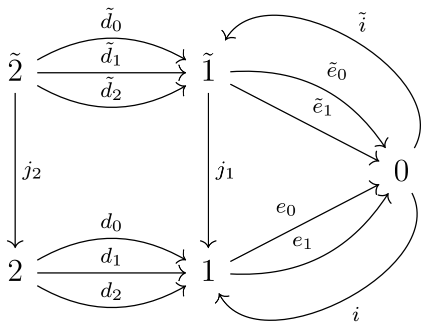
where the third isomorphism holds since F is a 2-sheaf. That these isomorphisms are natural follows because in each step it is natural. □

Definition 23. *Let $\mathcal{C}$ have all pullbacks. A **pretopology** or **basis for a topology** on $\mathcal{C}$ is a function which assigns to $U\in\mathcal{C}$ a set $Cov(U)$ consisting of* 1*-cells into U such that*

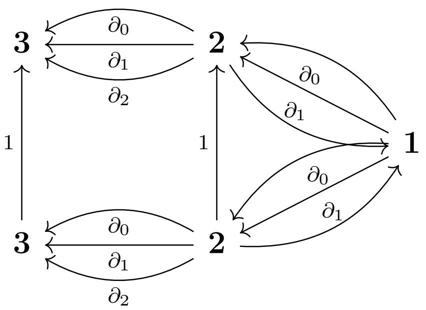
- *Stability axiom: For each $f\in\mathcal{C}(V,U)$ and $\{f_i:U_i\to U\}\in Cov(U)$, we have $\{f^\star U_i\to V\}\in Cov(V)$ (where $f^\star U_i\to V$ is the pullback of f_i along f).*
- *Identity cover: For each $U\in\mathcal{C}$, $\{Id_U\}\in Cov(U)$.*
- *Transitivity axiom: If $\{U_i\to U\}\in Cov(U)$ and for each i, $\{V_j^{(i)}\to U_i\}\in Cov(U_i)$, then is $\{V_j^{(i)}\to U_i\to U\}\in Cov(U)$.*

3.0.1 Characterization of sheaves

In this (sub)section we are going to give a characterization of a presheaf being a sheaf.
Consider again the category $\mathcal{D}$ given by:



subject to the *simplicial identities*. Let $N : \mathcal{D}^{op} \to \mathbf{Set}$ be the functor corresponding to the following diagram in **Set**:



where $\mathbf{i}$ $(i = 1, 2, 3)$ is a set $\{x_1, ..., x_i\}$ with i elements and

$$\partial_j : \mathbf{i} \to \mathbf{i+1} : x_k \mapsto x_{k+j(\bmod i+1)}.$$

Let $\mathcal{U}$ be a set, denote by $\lceil \mathcal{U} \rceil$ the functor $\mathbf{1_{Cat}} \to \mathbf{Set}\star \mapsto \mathcal{U}$ and let $\mathcal{D}_{\mathcal{U}}$ be the opposite of the comma category $N/\lceil \mathcal{U} \rceil$, i.e. the following diagram is a comma square:

$$\begin{array}{ccc} \mathcal{D}_{\mathcal{U}}^{op} & \longrightarrow & \mathbf{1_{Cat}} \\ \downarrow & & \downarrow \lceil \mathcal{U} \rceil \\ \mathcal{D}^{op} & \xrightarrow{N} & \mathbf{Set} \end{array}$$

So (by definition of the comma category) the objects of $\mathcal{D}_{\mathcal{U}}^{op}$ (so also $\mathcal{D}_{\mathcal{U}}$) are of the form (x, u) with $x \in \mathcal{D}$ an object and $u : N(x) \to \mathcal{U}$ and since $N(x)$ is a set with either $1, 2$ or 3 elements, u is given by a sequence of $1, 2$ or 3 elements in $\mathcal{U}$, i.e. a sequence of elements of $\mathcal{U}$ of length $N(x)$.
A morphism (x, u) to (y, v) (in $\mathcal{D}_{\mathcal{U}}^{op}$) is given by a morphism $f : x \to y$ in $\mathcal{D}^{op}$ such that $v \circ f = u$. So we have

$$\mathcal{D}_{\mathcal{U}}{}^{op}((x, u_0...u_x), (x, v_0...v_x)) = \begin{cases} \{Id_x^{op}\}, & \text{if } v_i = u_i \forall i = 0, ..., x \\ \emptyset, & \text{else} \end{cases}$$

$$\mathcal{D}_{\mathcal{U}}{}^{op}((0, u), (1, vw)) = \begin{cases} \{e_0^{op}, e_1^{op}\}, & \text{if } u = v = w \\ \{e_0^{op}\}, & \text{if } v = u, u \neq w \\ \{e_1^{op}\}, & \text{if } w = u, u \neq v \\ \emptyset, & \text{if } u \notin \{v, w\} \end{cases}$$

The same for when we replace 1 and e_i by $\tilde{1}$ and $\tilde{e}_i$.

$$\mathcal{D_U}^{op}((1, u_0u_1), (2, v_0v_1v_2)) = \begin{cases} \{d_0^{op}, d_1^{op}, d_2^{op}\}, & \text{if } u_i = v_j, \forall i, j \in \{0,1,2\} \\ \{d_i^{op}\}, & \text{if } d_i(u_j) = v_{i+j(\bmod 3)} \text{ but not all equal} \\ \emptyset, & \text{if } \{u_i, i = 0, 1\} \cap \{v_j, j = 0, 1, 2\} = \emptyset \end{cases}$$

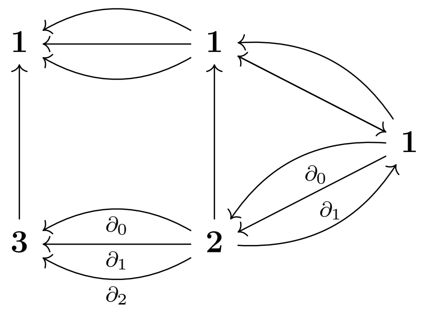
The same for when we replace 2 and d_i by $\tilde{2}$ and $\tilde{d}_i$.

$$\mathcal{D_U}^{op}((x, u_0...u_x), (x, v_0...v_x)) = \begin{cases} \{j_x^{op}\}, & \text{if } v_i = u_i \forall i = 0, ..., x \\ \emptyset, & \text{else} \end{cases}$$

And finally we also have

$$\mathcal{D_U}^{op}((1, vw), (0, u)) = \begin{cases} \{i^{op}\}, & \text{if } u = v = w \\ \emptyset, & \text{else} \end{cases}$$

The same for when we replace 1 and i by $\tilde{1}$ and $\tilde{i}$. So this describes $\mathcal{D_U}$ completely. The functor $\mathcal{D}_{\mathcal{U}}^{op} \to \mathcal{D}^{op}$ is the forgetfull functor, i.e. $(x, u) \mapsto x$.
Recall the functor $J : \mathcal{D}^{op} \to \mathbf{Cat}$ which corresponds with the following diagram (in $\mathbf{Cat}$):



The composite

$$\mathcal{D}_{\mathcal{U}}^{op} \to \mathcal{D}^{op} \to \mathbf{Cat},$$

is denoted by $J_{\mathcal{U}}$.
We are now going to construct a functor $S_{\mathcal{U}} : \mathcal{D_U} \to \mathcal{C}$: Let u, v, w be arrows into U. Let $\tilde{V}_{uv}$ (resp. V_{uv}) be the comma object (resp. pullback) of u along v:

$$\begin{array}{ccc} V_{uv} & \xrightarrow{d_v^{uv}} & V_v \\ \downarrow{\scriptstyle d_u^{uv}} & & \downarrow{\scriptstyle v} \\ V_u & \xrightarrow{u} & U \end{array} \qquad \begin{array}{ccc} \tilde{V}_{uv} & \xrightarrow{\tilde{d}_v^{uv}} & V_v \\ \downarrow{\scriptstyle \tilde{d}_u^{uv}} & & \downarrow{\scriptstyle v} \\ V_u & \xrightarrow{u} & U \end{array}$$

Let the following squares be pullback squares:

$$\begin{array}{ccc} V_{uvw} & \xrightarrow{d_{vw}^{uvw}} & V_{vw} \\ \downarrow{\scriptstyle d_{uv}^{uvw}} & & \downarrow{\scriptstyle d_w^{vw}} \\ V_{uv} & \xrightarrow{d_v^{uv}} & V_v \end{array} \qquad \begin{array}{ccc} \tilde{V}_{uvw} & \xrightarrow{\tilde{d}_{vw}^{uvw}} & \tilde{V}_{vw} \\ \downarrow{\scriptstyle \tilde{d}_{uv}^{uvw}} & & \downarrow{\scriptstyle \tilde{d}_w^{vw}} \\ \tilde{V}_{uv} & \xrightarrow{\tilde{d}_v^{uv}} & V_u \end{array}$$

Using the universal properties from the comma objects and the pullback, we have 1-cells $j_2 : \tilde{V}_{uvw} \to V_{uvw}$ and $j_1 : \tilde{V}_{uv} \to V_{uv}$.

Lemma 14. *Let $\mathcal{U}$ be a set of arrows into an object U of a finitely complete 2-category. The following data defines a functor $S_{\mathcal{U}} : \mathcal{D}_{\mathcal{U}} \to \mathcal{C}$:*

$$
\begin{array}{ccc}
(2,(u,v,w)) & \xleftarrow[j_2]{} & (\tilde{2},(u,v,w)) \\
\downarrow{\scriptstyle d_0} & & \downarrow{\scriptstyle d_0} \\
(1,(u,v)) & \xleftarrow[j_1]{} & (\tilde{1},(u,v)) \\
\downarrow{\scriptstyle d_0} & \swarrow{\scriptstyle d_0} & \\
(0,(u)) & &
\end{array}
\quad \overset{S_{\mathcal{U}}}{\Longrightarrow} \quad
\begin{array}{ccc}
V_{uvw} & \xleftarrow[j_2]{} & \tilde{V}_{uvw} \\
\downarrow{\scriptstyle d^{uvw}_{uv}} & & \downarrow{\scriptstyle \tilde{d}^{uvw}_{uv}} \\
V_{uv} & \xleftarrow[j_1]{} & \tilde{V}_{uv} \\
\downarrow{\scriptstyle d^{uv}_{u}} & \swarrow{\scriptstyle \tilde{d}^{uv}_{u}} & \\
V_u & &
\end{array}
$$

Proof. This is clearly functorial by definition of the V_i's and $\tilde{V}_i$'s and the corresponding morphisms d_i and $\tilde{d}_i$. □

Lemma 15. *The following data defines a 2-natural transformation $\kappa = \kappa_{\mathcal{U}} : J_{\mathcal{U}} \to \mathcal{C}(S_{\mathcal{U}}, U)$:*

$$
\begin{aligned}
\kappa_{(0,u)} :&\quad \mathbf{1} \to \mathcal{C}(V_u, U) : &\star_1 \mapsto u \\
\kappa_{(1,uv)} :&\quad \mathbf{1} \to \mathcal{C}(V_{uv}, U) : &\star_1 \mapsto u \circ d^{uv}_u \\
\kappa_{(2,uvw)} :&\quad \mathbf{1} \to \mathcal{C}(V_{uvw}, U) : &\star_1 \mapsto u \circ d^{uv}_u \circ d^{uvw}_{uv}
\end{aligned}
$$

and

$$\kappa_{(\tilde{1},uv)} : \mathbf{2} \to \mathcal{C}(\tilde{V}_{uv}, U),$$

maps $f : \star_1 \to \star_2$ to the 2-cell $\lambda : u \circ \tilde{d}^{uv}_u \to v \circ \tilde{d}^{uv}_v$ which is given by the universal property of $\tilde{V}_{uv}$ being the comma object of u along v. And

$$\kappa_{(\tilde{2},uvw)} : \mathbf{3} \to \mathcal{C}(\tilde{V}_{uvw}, U) : (f : \star_1 \to \star_2)$$

maps $\star_1 \xrightarrow{f} \star_2 \xrightarrow{g}$ is mapped to λ composed with the identity 2-cell $\tilde{d}^{uv}_v \circ \tilde{d}^{uvw}_{uv} = \tilde{d}^{vw}_v \circ \tilde{d}^{uvw}_{vw}$.

Proof. This is again clear by definition of the V_i's and $\tilde{V}_i$'s and the corresponding morphisms d_i and $\tilde{d}_i$. □

Proposition 8. *Let $\mathcal{C}$ be a finitely complete 2-site, $U \in \mathcal{C}$ and $\mathcal{U}$ a set of arrows into U. Let R be the U-crible generated by $\mathcal{U}$. Then is R a left Kan extension of $J_{\mathcal{U}}$ along $S_{\mathcal{U}}$, i.e. $\kappa_{\mathcal{U}}$ induces an isomorphism of categories*

$$[\mathcal{C}^{op}, \mathbf{Cat}](R, F) \cong [\mathcal{D}^{op}_{\mathcal{U}}, \mathbf{Cat}](J_{\mathcal{U}}, FS_{\mathcal{U}}).$$

The isomorphism of the previous proposition is given as follows: Let $F : \mathcal{C}^{op} \to \mathbf{Cat}$ be a 2-functor and $\alpha \in [\mathcal{C}^{op}, \mathbf{Cat}](R, F)$ a 2-natural transformation. We have to define a 2-natural transformation $J_{\mathcal{U}} \to FS_{\mathcal{U}}$. Define this as

$$(J_{\mathcal{U}})_{(x,u)} \xrightarrow{\kappa_{(x,u)}} \mathcal{C}(S_{\mathcal{U}}, U)_{(x,u)} \xrightarrow{\alpha_{S_{\mathcal{U}}(x,u)}} (FS_{\mathcal{U}})_{(x,u)}\,,$$

for $(x,u) \in \mathcal{D}_{\mathcal{U}}^{op}$. Notice that this is well-defined because RV consists by definition of those arrows $V \to U$ which factor through some arrow in $\mathcal{U}$ and by definition of $\mathcal{K}_{(x,u)}$ we have that all 1-and 2-cells factor through u which lies in $\mathcal{U}$. Since it is a composition of 2-natural transformations, it is clearly a 2-natural transformation.

Proposition 9. *Suppose $\mathcal{C}$ is finitely complete and with a topology. A 2-functor $F : \mathcal{C}^{op} \to \mathcal{K}$ is a $\mathcal{K}$-valued 2-sheaf if and only if for each 0-cell $U \in \mathcal{C}$ and covers $\mathcal{U}$ of U, the 2-natural transformation*

$$J_{\mathcal{U}} \xrightarrow{\kappa_{\mathcal{U}}} \mathcal{C}(S_{\mathcal{U}}, U) \xrightarrow{F} \mathcal{K}(FU, FS_{\mathcal{U}}),$$

exhibits FU as a $J_{\mathcal{U}}$-weighted limit for $FS_{\mathcal{U}}$.

Proof. It suffices to show the statement for $\mathcal{K} = \mathbf{Cat}$ because a 2-functor $F : \mathcal{C}^{op} \to \mathcal{K}$ is a $\mathcal{K}$-valued 2-sheaf if and only if $\mathcal{K}(X, F) : \mathcal{C}^{op} \to \mathbf{Cat}$ is a 2-sheaf for each $X \in \mathcal{K}$ and weighted limits are defined representable.
Let R be a U-crible and let $\mathcal{U}$ consists of the arrows which R represents, then by the previous proposition we have $[\mathcal{C}^{op}, \mathbf{Cat}](R, F) \cong [\mathcal{D}_{\mathcal{U}}^{op}, \mathbf{Cat}](J_{\mathcal{U}}, FS_{\mathcal{U}})$. But in the proof that $\mathbf{Cat}$ is complete, we have shown $[\mathcal{D}_{\mathcal{U}}^{op}, \mathbf{Cat}](J_{\mathcal{U}}, FS_{\mathcal{U}}) \cong \lim(J_{\mathcal{U}}, FS_{\mathcal{U}})$.
On the other hand, the Yoneda lemma gives us $[\mathcal{C}^{op}, \mathbf{Cat}](\mathcal{C}(-, U), F) \cong F(U)$. So F is a 2-sheaf if and only if $F(U) \cong \lim(J_{\mathcal{U}}, FS_{\mathcal{U}})$. □

3.0.2 Sheafification

Let $(\mathcal{C}, \mathcal{T})$ be a 2-site (with small hom-categories). In this (sub)section we are going to show that the inclusion 2-functor $Sh(\mathcal{C}, \mathbf{Cat}) \to Fun_2(\mathcal{C}^{op}, \mathbf{Cat})$ has a left adjoint Σ which is (left) exact. So this means precisely that $Sh(\mathcal{C}, \mathbf{Cat})$ is a localisation of the presheaf category $Pr_2(\mathcal{C}) := Fun_2(\mathcal{C}^{op}, \mathbf{Cat})$.
We are first going to define a 2-functor:

$$L : Pr_2(\mathcal{C}) \to Pr_2(\mathcal{C}).$$

Let $P \in Pr(\mathcal{C})$ be a presheaf and $X \in \mathcal{C}$ a 0-cell. Define

$$(LP)X := colim_{R \in \mathcal{T}(X)} Pr(\mathcal{C})(R, P) = colim \left(\mathcal{T}(X)^{op} \to Pr(\mathcal{C}) \xrightarrow{Pr(\mathcal{C})(-,P)} \mathbf{Cat} \right).$$

Since $\mathcal{T}(X)^{op}$ is a 1-category considered as a 2-category, we have that this colimit is an *ordinary colimit* (i.e. no conditions on the 2-cells). In particular we have for each $R \in \mathcal{T}(X)$ the following morphism:

$$s_R^P : Pr_2(\mathcal{C})(R, P) \to LP(X).$$

This defines $LP : \mathcal{C}^{op} \to \mathbf{Cat}$ on the 0-cells. We now define it for the 1-cells, so let $f \in \mathcal{C}(Y, X)$ be a 1-cell. Let $R \in \mathcal{T}(X)$ be a covering sieve and consider (again) the following pullback square:

$$\begin{array}{ccc} R_f & \xrightarrow{f_R} & R \\ \downarrow & & \downarrow r \\ \mathcal{C}(-, Y) & \xrightarrow{f \circ -} & \mathcal{C}(-, X) \end{array}$$

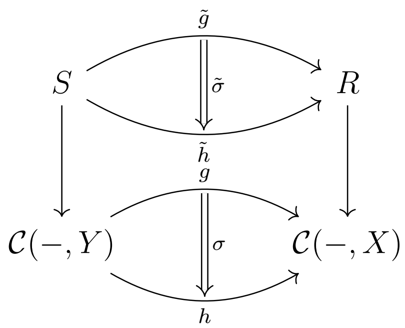
Consider:

$$Pr_2(R,P) \xrightarrow{Pr_2(f_R,P)} Pr_2(R_f,P) \xrightarrow{s^P_{R_f}} LP(Y).$$

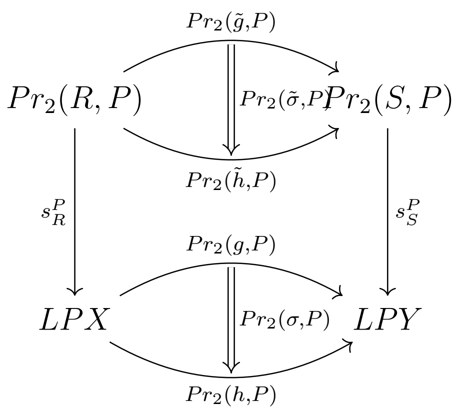
Using the colimit property of $LP(X)$, there exists a unique $LP(f) : LP(X) \to LP(Y)$ such that the following diagram commutes:

$$\begin{array}{ccc} Pr_2(R,P) & \xrightarrow{Pr_2(f_R,P)} & Pr_2(R_f,P) \\ {\scriptstyle s^P_R}\downarrow & & \downarrow{\scriptstyle s^P_{R_f}} \\ LP(X) & \xrightarrow{LP(f)} & LP(Y) \end{array}$$

We now define LP on the 2-cells: Let $\sigma : f \implies g : Y \to X$ be a 2-cell in $\mathcal{C}$. Define $(LP)\sigma$ to be the 2-cell which satisfies the following property: For all covering cribles $R \in \mathcal{T}(X)$ and $S \in \mathcal{T}(Y)$, if



commutes, then should the following diagram commute:



Notice that $LP(\sigma)$ exists (and is unique) since this is also given by the universal property of the colimit $LP(X)$. The difference between defining $LP(f)$ and $LP(\sigma)$ lies in the fact that we can not take the pullback over a 2-cell, hence we consider the factorization over every 2-cell.
That the 2-functor L is left exact (i.e. preserves finite limits) follows since the colimits are filtered and since a limit in a presheaf category is computed object-wise.

Definition 24. *The assignment $P \mapsto LP$ defines a 2-natural transformation*

$$l : Id_{Pr(\mathcal{C})} \implies L.$$

We say that $P \in Pr_2(\mathcal{C})$ is 1-separated if

$$Pr_2(\mathcal{C})(i, P) : Pr_2(\mathcal{C})(\mathcal{C}(-, X), P) \to Pr_2(\mathcal{C})(R, P),$$

is injective on objects and faithfull for all covering cribles $i : R \to \mathcal{C}(-, X)$ in $\mathcal{T}(X)$.

Lemma 16. *For $P \in Pr_2(\mathcal{C})$ a presheaf, LP is 1-separated.*

Proof. We first show that for (each) $P \in Pr_2(\mathcal{C})$ and $i : R \to \mathcal{C}(-, X) \in \mathcal{T}(X)$ the functor

$$Pr_2(\mathcal{C})(i, LP) : Pr_2(\mathcal{C})(\mathcal{C}(-, X), LP) \to Pr_2(\mathcal{C})(R, LP),$$

is injective on objects.
Let $f, g \in Pr_2(\mathcal{C})(\mathcal{C}(-, X), LP)$ such that $f \circ i = g \circ i$. By Yoneda we have an equivalence of categories

$$\phi_P^X : Pr_2(\mathcal{C})(\mathcal{C}(-, X), P) \cong P(X).$$

Since $(LP)X = colim_{\tilde{R} \in \mathcal{T}(C)} Pr_2(\mathcal{C})(\tilde{R}, P)$, $\phi_{LP}^X(f)$ and $\phi_{LP}^X(g)$ are represented by $u : R_1 \to P$ and $v : R_2 \to P$ for some $R_1, R_2 \in \mathcal{T}(C)$ and we can assume $R_1 \cong R_2 \subseteq R$ (since u and $u|_{R_1 \cap R_2 \cap R_3}$ are equal in $(LP)X$).
We now claim that for each $s \in R_1$ (so $s : Y \to X$ for some Y), its image under the composite

$$R_1 Y \subseteq RY \xrightarrow{i_Y} \mathcal{C}(Y, X) \xrightarrow{f_Y} (LP)Y,$$

is (represented by) $u \circ \tilde{s}$ (and if we replace f_Y by g_Y we get $v \circ \tilde{s}$) where $\tilde{s} := \phi^{-1}(s)$. So in particular we have $i \circ \tilde{s} = s$.
The naturality of the Yoneda lemma gives us the following commuting diagrams:

$$\begin{array}{ccc} \mathcal{P}(\mathcal{C}(-, Y), R) & \xrightarrow{\phi} & RY \\ \downarrow{\scriptstyle i\circ -} & & \downarrow{\scriptstyle i_Y} \\ \mathcal{P}(\mathcal{C}(-, Y), \mathcal{C}(-, X)) & \xrightarrow{\phi} & \mathcal{C}(Y, X) \\ \downarrow{\scriptstyle f\circ -} & & \downarrow{\scriptstyle f_Y} \\ \mathcal{P}(\mathcal{C}(-, Y), LP) & \xrightarrow{\phi} & (LP)Y \end{array} \qquad \begin{array}{ccc} \mathcal{P}(\mathcal{C}(-, X), LP) & \xrightarrow{\phi} & (LP)X \\ \downarrow{\scriptstyle -\circ s} & & \downarrow{\scriptstyle (LP)(s)} \\ \mathcal{P}(\mathcal{C}(-, Y), LP) & \xrightarrow{\phi} & (LP)Y \end{array}$$

Thus

$$f_Y \circ i_Y(s) = \phi_{LP}^Y(f \circ i \circ \tilde{s}) = \phi_{LP}^Y(f \circ s) = (LP)(s)(\phi_{LP}^X(f)) = (LP)(s)(u).$$

We now calculate $(LP)(s)(u)$. It is (by definition) the unique morphism such that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{P}(R, P) & \xrightarrow{s_R^P} & (LP)X \\ \downarrow{\scriptstyle -\circ s_R} & & \downarrow{\scriptstyle (LP)(s)} \\ \mathcal{P}(R_s, P) & \xrightarrow{s_{R_s}^P} & (LP)Y \end{array}$$

where R_s is the pullback of R along $\mathcal{C}(-,s)$. Thus (abusing notation)

$$(LP)(s)(u) = LP(s)(s^P_R(u)) = s^P_{R_f}(u \circ s_R) = s^P_{R_s}(u \circ \tilde{s} \circ r_s,$$

i.e. $(LP)(s)(u)$ is represented by $u \circ \tilde{s} \circ r_s$. But $r_s : R_s \to \mathcal{C}(-,Y) \in \mathcal{T}(Y)$, thus $u \circ \tilde{s} \circ r_s$ and $u \circ \tilde{s}$ represent the same element in $(LP)Y$, thus we indeed have shown the claim.
Since $f \circ i = g \circ i$, we have

$$u \circ \tilde{s} = f_Y \circ i_Y(s) = g_Y \circ i_Y(s) = v \circ \tilde{s}.$$

Since this equality hold in $(LP)Y$ (for each $s \in R_1$), there exists some covering crible $S_s \in \mathcal{T}(Y)$ such that

$$\left(S_s \hookrightarrow \mathcal{C}(-,Y) \xrightarrow{\tilde{s}} R \xrightarrow{u} P\right) = \left(S_s \hookrightarrow \mathcal{C}(-,Y) \xrightarrow{\tilde{s}} R \xrightarrow{v} P\right).$$

Let $T := \{s \circ t | s \in R_1, t \in S_s\}$. This is a covering sieve of X (using the 3^{th} axiom of a pretopology since $S_s \in \mathcal{T}(Y), R_1 \in \mathcal{T}(X)$). But $u|_T = v|_T$, so u and v represent the same element in $(LP)X$ which shows that $f = \phi^{-1}(u) = \phi^{-1}(v) = g$. Thus we indeed have injectiveness on objects.
In particular we have that $\mathcal{P}(i, L(2 \pitchfork P))$ is injective on objects. Since L is left exact, it preserves cotensors and using the definition of the cotensor, we get that the functor

$$\mathcal{P}(i, L(2 \pitchfork P)) : \mathcal{P}(\mathcal{C}(-,X), L(2 \pitchfork P)) \to \mathcal{P}(R, L(2 \pitchfork P)),$$

becomes

$$\mathbf{Cat}(2, \mathcal{P}(\mathcal{C}(-,X), LP)) \to \mathbf{Cat}(2, \mathcal{P}(R, LP))$$

which is (completely) determined by sending a 2-cell α to $\alpha \circ i$. Thus injectiveness of $\mathcal{P}(i, L(2 \pitchfork P))$ on objects means precisely that $\mathcal{P}(i, LP)$ is faithfull. Thus LP is indeed 1-separated. □

Lemma 17. *$P \in Pr_2(\mathcal{C})$ is 1-separated if and only if $l_P : P \to LP$ is a monomorphism.*

We say that $P \in Pr_2(\mathcal{C})$ is 2-separated if

$$Pr_2(\mathcal{C})(i,P) : Pr_2(\mathcal{C})(X,P) \to Pr_2(\mathcal{C})(R,P),$$

is chronic for all covering cribles $i : R \to X$ in $\mathcal{T}(X)$, i.e. it is 1-separated and $Pr_2(\mathcal{C})(i,P)$ is moreover full.

Lemma 18. *If $P \in Pr_2(\mathcal{C})$ is 1-separated, LP is 2-separated.*

Lemma 19. *$P \in Pr_2(\mathcal{C})$ is 2-separated if and only if $l_P : P \to LP$ is chronic.*

Lemma 20. *If $P \in Pr_2(\mathcal{C})$ is 2-separated, LP is a sheaf.*

Proof. Since P is 2-separated, it remains to show that if $R \overset{i}{\hookrightarrow} \mathcal{C}(-,X) \in \mathcal{T}(X)$ is a covering, then is

$$Pr_2(\mathcal{C})(i,P) : Pr_2(\mathcal{C})(\mathcal{C}(-,X), LP) \to Pr_2(\mathcal{C})(R, LP),$$

surjective on objects. Let $\alpha : R \to LP$ be a 2-natural transformation. We have to show that it extends to a (unique) 2-natural transformation $\beta : \mathcal{C}(-,X) \to LP$. By Yoneda we have that β corresponds with a unique object $X_\beta \in LPX$. We now show how X_β is constructed:
Form the following pullback square:

$$\begin{array}{ccc} S & \longrightarrow & R \\ \downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle \alpha} \\ P & \xrightarrow{l_P} & LP \end{array}$$

So this defines a 2-natural transformation $\mu : S \to P$. We then define x_β as the image of μ under

$$s_S^P : Nat_2(S,P) \to LPX,$$

i.e. $x_\beta := s_S^P(\mu)$. Notice that s_S^P is only defined when $S \in \mathcal{T}(X)$: By the last axiom of a Grothendieck topology, we can conclude $S \in \mathcal{T}(X)$ if for any $Y \in \mathcal{C}$ and any $f : Y \to X \in RX$ we have $S_f \in \mathcal{T}(D)$, where S_f is defined by the following pullback square(s):

$$\begin{array}{ccccc} S_f & \longrightarrow & R_f & \longrightarrow & \mathcal{C}(-,Y) \\ \downarrow{\scriptstyle f_S} & & \downarrow{\scriptstyle f_R} & & \downarrow{\scriptstyle f\circ -} \\ S & \longrightarrow & R & \xrightarrow{i} & \mathcal{C}(-,X) \end{array}$$

□

Lemma 21. *$P \in Pr_2(\mathcal{C})$ is a sheaf if and only if $l_P : P \to LP$ is an isomorphism.*

Corollary 2. *The functor $\Sigma := L^3$ is the left adjoint of the inclusion $Sh_2(\mathcal{C}, \mathcal{T}) \to Pr_2(\mathcal{C})$ and it is left exact.*

Theorem 3. *("**Comparison lemma**")Let $\mathcal{E}$ be a 2-site with small hom-categories. Let $\mathcal{C}$ be a small 2-category such that there is a fully faithfull 2-functor $\mathcal{J} : \mathcal{C} \to \mathcal{E}$ such that for each object $X \in \mathcal{E}$, there exists a set $\mathcal{U} \in Cov(X)$ for which the source of each arrow in $\mathcal{U}$ is in the image of $\mathcal{J}$.*
*Let $\mathcal{C}$ have the largest topology such that $F\mathcal{J}^{op} : \mathcal{C}^{op} \to$ **Cat** is a 2-sheaf for all 2-sheaves F on $\mathcal{E}$. Then $\mathcal{J}$ induces an equivalence of 2-categories*

$$Sh(\mathcal{E}, \mathbf{Cat}) \cong Sh(\mathcal{C}, \mathbf{Cat}).$$

3.0.3 Acute sets

Definition 25. *Let $\mathcal{C}$ be a finitely complete 2-category and U a 0-cell. A set $\mathcal{U}$ of 1-cells into U is **acute** if for each chronic $m : V \to U$ we have that if for each $f : W \to U \in \mathcal{U}$, f factors through m, then m is an isomorphism.*

Proposition 10. *Assume $\mathcal{C}$ has all small coproducts. Let $U \in \mathcal{C}$ be a* 0*-cell and let $\mathcal{U}$ be a set of* 1*-cells. Then is $\mathcal{U}$ acute if and only if the singleton set containing*

$$\left(\coprod_{f_V : V \to U \in \mathcal{U}} V\right) \xrightarrow{\coprod_{f_V \in \mathcal{U}} f_V} V, \tag{3.1}$$

is acute.

Proof. Denote by $f : \tilde{V} \to U$ the 1-cell (3.1) and let $i_V : V \to \tilde{V}$ be the canonical 1-cell. Assume $\mathcal{U}$ is acute. We have to show that $\{f\}$ is acute. So assume $m : W \to U$ is a chronic such that $f = m \circ e$ for some 1-cell $e : \tilde{V} \to W$. Thus for each $f_V \in \mathcal{U}$, we also have

$$f_V = f \circ i_V = m \circ e \circ i_V.$$

Since each $f_V \in \mathcal{U}$ factors through m, we conclude by acuteness of $\mathcal{U}$ that m is an isomorphism.
Conversely assume $\{f\}$ is acute. We have to show that $\mathcal{U}$ is acute. So assume $m : W \to U$ is a chronic such that for each $f_V \in \mathcal{U}$, $f_V = m \circ e_V$ for some 1-cell $e : \tilde{V} \to W$. Since $\tilde{V}$ is the coproduct, we have that there exists a unique 1-cell $g : \tilde{V} \to W$ such that $e_V = g \circ i_V$. By uniqueness of the 1-cell $f : \tilde{V} \to U$, we also have $f = m \circ g$. Thus by acuteness of $\{f\}$, we conclude that m is indeed an isomorphism. □

Example 18. *A singleton set $\{f : W \to U\}$ is acute if and only if f is acute.*

Definition 26. *A set $\mathcal{G}$ of objects of $\mathcal{C}$ is **acutely generating** if for each object $U \in \mathcal{C}$, the set of all arrows in U with domain in $\mathcal{G}$ is acute.*

Lemma 22. *Let $\mathcal{C}$ be a finitely complete* 2*-category (with small hom-categories) and $\mathcal{G}$ an acutely generating set of $\mathcal{C}$. The chronic subobjects of a given objects form a (small) set.*
Consequently, such a 2*-category contains a small full sub-*2*-category which is closed under finite limits and chronic subobjects which contains an acutely generating set of objects.*

Proof. Fix $U \in \mathcal{C}$ and let M be the set consisting of all arrows $G \to U$ with $G \in \mathcal{G}$ (M is indeed a set since $\mathcal{G}$ is a set and $\mathcal{C}$ has small hom-categories). For a chronic $m : U_1 \to U$, let $N(m)$ be the subset of M containing those arrows $G \to U$ which factor through m.
We show that if $m_1 : U_1 \to U$ and $m_2 : U_2 \to U$ are non-equivalent chronic subobjects, then $N(m_1) \neq N(m_2)$. Consider the following pullback diagram:

$$\begin{array}{ccc} U_{12} & \xrightarrow{n_2} & U_2 \\ \downarrow{\scriptstyle n_1} & & \downarrow{\scriptstyle m_2} \\ U_1 & \xrightarrow{m_1} & U \end{array}$$

If m_1 is non-equivalent to m_2, then (by definition of equivalence of subobjects) we necessarily have that n_1 and n_2 can't be both isomorphisms. So we can assume that n_2 is not iso. Notice that n_2 is chronic since it the pullback of m_2 a chronic.

Since $\mathcal{G}$ is acutely generating, M is acute and thus (since n_2 is chronic but not an iso) there exists some $G \in \mathcal{G}$ and $f : G \to U_2$ such that n_2 does not factor through f.
We now claim that $m_2 \circ f \in N(m_2) \setminus N(m_1)$. Clearly $m_2 \circ f \in N(m_2)$, now assume that $m_2 \circ f \in N(m_1)$, thus there exists some $g : G \to U_1$ such that $m_2 \circ f = m_1 \circ g$. So by the universal property of the pullback U_{12}, there exists some (unique) arrow $h : G \to U_{12}$ such that $f = n_2 \circ n$, but this contradicts the assumption on f.
So we indeed conclude that $N(m_1) \neq N(m_2)$. Thus each chronic subobject is represented by a unique subset of M, i.e. the collection of chronic subobjects can be identified as a subcollection of the powerset of M which then shows the first part of the lemma because M is a set.
For the second part, let $\mathcal{C}_0$ be the full sub-2-category generated by $\mathcal{G}$, so $\mathcal{C}_0$ is small. Inductively define $\mathcal{D}_{i+1}$ ($i \geq 0$) as the full sub-2-category of $\mathcal{C}$ generated by $\mathcal{C}_i$ by adding the finite limits. Then define $\mathcal{C}_{i+1}$ as the full sub-2-category generated by $\mathcal{D}_{i+1}$ by adding the chronic subobjects).
Let $\tilde{\mathcal{C}} := \bigcup_{i>0} \mathcal{C}_i$. This sub-2-category is finitely complete, contains all chronic subobjects by construction. Since each $\mathcal{C}_i$ is small, so is $\tilde{\mathcal{C}}$ since the union of sets is again a set. □

Definition 27. *The* ***canonical topology*** *on a* 2*-category is the largest topology for which all the representable* 2*-functors are* 2*-sheaves. The sheaves for this topology are called the* ***canonical* 2*-sheaves***.

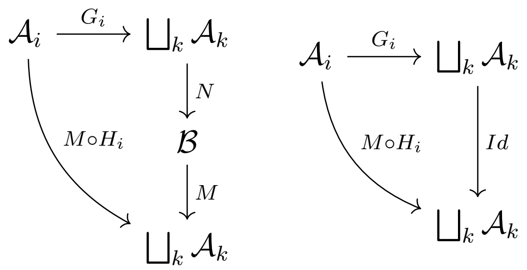
Proposition 11. *Let* $\mathcal{C}$ *be a finitely complete* 2*-site. If all representables are* 2*-sheaves, then all covering families are acute.*

Proof. Let $\mathcal{U} \in Cov(U)$. By the characterisation of 2-sheaves and since $\mathcal{C}(-, U)$ is a 2-sheaf $FU := \mathcal{C}(U, U)$ is a $J_{\mathcal{U}}$-indexed limit for $FS_{\mathcal{U}} := \mathcal{C}(S_{\mathcal{U}}, U)$.
Let $m : V \to U$ be chronic such for each $f : W \to U \in \mathcal{U}$, f factors through m. But $\mathcal{C}(-, U)$ is a colimit, therefore we have that m has a right inverse and consequently is an isomorphism. □

Example 19.

- *A set of arrows with common target in* **Cat** *is acute if all the arrows are jointly surjective on objects.*
- *A set of arrows with common target in* **Cat** *factors into an acute set followed by a chronic arrow.*

Proof. Let $\{F_I : \mathcal{A}_i \to \mathcal{A}\}$ be a set in **Cat** which is jointly surjective, i.e. $\bigsqcup_i F_i : \bigsqcup_i \mathcal{A}_i \to \mathcal{A}$ is essentially surjective. Let $P : \mathcal{B} \to \mathcal{A}$ be chronic such that for each i, there exists some $G_i : \mathcal{A}_i \to \mathcal{B}$ such that $M \circ G_i = F_i$. Let $A \in \mathcal{A}$. Since $\bigsqcup_i F_i$ is surjective, there exists some i such that $F_i(A_i) = A$, thus $A = F_i(A_i) = M(G_i(A_i))$. Thus M is surjective on objects and thus by chronicness of M is an isomorphism, thus $\{F_i\}$ is acute.
Consider a set $\{F_i : \mathcal{A}_i \to \mathcal{A}\}$ of functors. This set induces an acute set $\{G_i : \mathcal{A}_i \to \bigsqcup_k \mathcal{A}_k\}$ and a chronic $\bigsqcup_k \mathcal{A}_k \to \mathcal{A}$, indeed: To show acuteness, assume there is some chronic $M : \mathcal{B} \to \bigsqcup_k \mathcal{A}_k$ for which each G_i factors through by some $H_i : \mathcal{A}_i \to \mathcal{B}$. So by the universal property of the coproduct, there exists some $N : \bigsqcup_k \mathcal{A}_k \to \mathcal{B}$ such that $N \circ G_i = H_i$. Since the following commuting diagrams:



We have by uniqueness of the universal property of the coproduct we have that $M \circ N = Id$, thus M is a (split) epimorphism and thus by chronicness an isomorphism. The chronicness of $F : \bigsqcup_k \mathcal{A}_k \to \mathcal{A}$ is immediate by the uniqueness of factorization.

□

3.0.4 Lex-total categories

Definition 28. *A* 2*-category* $\mathcal{K}$ *is* ***lex-total*** *when it has small hom-categories and the Yoneda embedding* $y : \mathcal{K} \to [\mathcal{K}^{op}, \mathbf{Cat}]$ *has a left adjoint which preserves finite limits.*

Theorem 4. *Every* 2*-topos* $\mathcal{K}$ *is lex-total.*

Proof. Assume $\mathcal{K} = Sh(\mathcal{C}, \mathbf{Cat})$ and let $i : \mathcal{K} \to Fun_2(\mathcal{C}^{op}, \mathbf{Cat})$ be the embedding. Then is the Yoneda embedding $\mathcal{K} \to Fun_2(\mathcal{K}^{op}, \mathbf{Cat})$ factorized as:

$$\mathcal{K} \xrightarrow{i} Fun_2(\mathcal{C}^{op}, \mathbf{Cat}) \xrightarrow{y} Fun_2(Fun_2(\mathcal{C}^{op}, \mathbf{Cat})^{op}, \mathbf{Cat}) \xrightarrow{Func_2(i^{op}, Id)} Fun_2(\mathcal{K}^{op}, \mathbf{Cat}).$$

By sheafification, i has a left-exact left adjoint and thus also $[i^{op}, Id]$. The Yoneda-embedding of the presheaf category has a left-exact left adjoint given by $Z(P)(U) = P(\mathcal{C}(-, U))$. So by composition, it follows that the Yoneda embedding of $\mathcal{K}$ also has a left exact left adjoint. □

Lemma 23. *The canonical topology on a lex-total* 2*-category* $\mathcal{K}$ *consists of the acute sets as covers.*

Proposition 12. *Every* $\mathbf{Cat}$*-valued canonical* 2*-sheaf on a lex-total* 2*-category* $\mathcal{K}$ *is representable.*

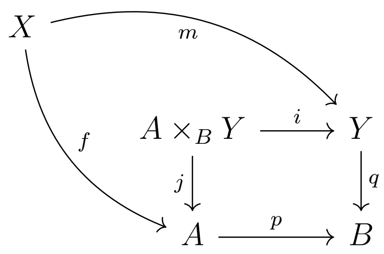
Lemma 24. *In a lex-total* 2*-category, the pushout of a chronic is a pullback, i.e. let* $m : X \to Y$ *be a chronic and consider its pushout along* $f : X \to A$*:*

$$\begin{array}{ccc} X & \xrightarrow{m} & Y \\ \downarrow{\scriptstyle f} & & \downarrow \\ A & \longrightarrow & B \end{array}$$

Then is this pushout square a pullback square.

Proof. First note that we can compute finite limits and colimits in $[\mathcal{K}^{op}, \mathbf{Cat}]$, because we can take the limit (resp. colimit) under the Yoneda embedding and then apply the left adjoint Z of the Yoneda embedding y to get the desired limit (resp.

colimit) since Z preserves finite limits by definition of lex-totalness (resp. Z preserves all colimits as a left adjoint). But (co)limits in the 2-functor category are computed object-wise. Thus it suffices to show the statement for $\mathcal{K} = \mathbf{Cat}$.
Consider the following diagram:



Here is B the pushout of m along f and $A \times_B Y$ is the pullback of p along q. Let $h : X \to A \times_B Y$ be the unique morphism induced by the universal property of the pullback. Each object of the pullback $A \times_B Y$ is given by some (a, y) with $a \in A, y \in Y$ such that $p(a) = q(y)$ and the canonical morphism h is given by $h(x) = (f(x), m(x))$. We now show that h is an isomorphism, to do this, we use the explicit description of the pushout in categories found in ([5]). In this paper the pushout along a fully faithfull functor is given (which we can apply since m is a chronic) which is moreover replete[2]. In order to get the repleteness, we assume Y is a skeleton. The objects in the pushout P then consists of the objects of C and the objects of Y without the objects of X (here we use that m is chronic, thus we consider X as a full subcategory of Y).
Let $(a, y) \in A \times_B Y$. Then we have that a is the image of a unique object $x \in X$. This can then be extended to the needed functor. □

We call a collection moderate if it has a size not greater that $Ob(\mathbf{Set})$:

Lemma 25. *Let $\mathcal{K}$ be a lex-total 2-category which has a moderate set of objects such that for each $X \in \mathcal{K}$, there exists an acute arrow $M \to X$ with $M \in \mathcal{M}$. Then has $\mathcal{K}$ an acutely generating set of objects.*

Proof. Assume it does not have an acutely generating set of objects. In particular we have that $\mathcal{K}$ is not small. We assume that $\mathcal{K}$ is skeletal so that we can order the objects of $\mathcal{K}$.
We now claim that we can order the objects in M such that for each $A \in \mathcal{M}$, we have that $\{B \in \mathcal{M} | B \leq A\}$ is small: Since M is moderate, we can assign to each object $B \in M$ a set $\tilde{B}$. So we can order $\{\tilde{B} | B \in M\}$ with the inclusion. Since the subsets of a set forms a set, we have that $\{\tilde{B} \leq \tilde{A} | B \in M\}$ is a set. So

$$B \leq A \iff \tilde{B} \subseteq \tilde{A}$$

defines a wanted ordering on the objects of M which shows the claim.
Since $\mathcal{K}$ has no acutely generating set, we have that these sets can't acutely generate $\mathcal{K}$. So this means that for each $A \in \mathcal{K}$, there exists a chronic $m_A : X_A \to Y_A$ which is not an isomorphism, but we have that each morphism $B \to X_A$ factors through m_A where $B \leq A$, i.e.

$$\mathcal{K}(B, m_A) : \mathcal{K}(B, X_A) \to \mathcal{K}(B, Y_A),$$

[2] A functor is replete if its image is replete, i.e. closed under isomorphisms.

is an isomorphism for all $B \leq A$.
For each $A \in \mathcal{K}$, consider the following 2-pushout (in $Fun_2(\mathcal{K}^{op}, \mathbf{Cat})$):

$$\begin{array}{ccc} \mathcal{K}(-, X_A) & \xrightarrow{\mathcal{K}(-,m_A)} & \mathcal{K}(-, Y_A) \\ \downarrow & & \downarrow{\scriptstyle \gamma_A} \\ \mathbf{1} & \xrightarrow{\quad \omega \quad} & P^{(A)} \end{array}$$

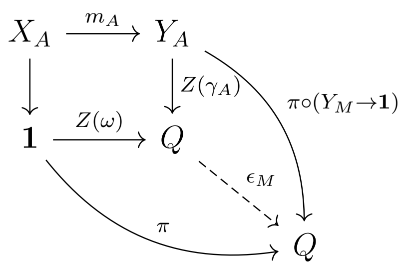
If we apply Z (the left adjoint of the Yoneda embedding), we get that the following diagram is a 2-pushout square since Z is left exact:

$$\begin{array}{ccc} Z(\mathcal{K}(-, X_A)) & \xrightarrow{Z(\mathcal{K}(-,m_A))} & Z(\mathcal{K}(-, Y_A)) \\ \downarrow & & \downarrow{\scriptstyle Z(\gamma_A)} \\ Z(\mathbf{1}) & \xrightarrow{\quad Z(\omega) \quad} & Z(P^{(A)}) \end{array}$$

Since the Yoneda embedding is fully faithfull and is a right adjoint, we have that the counit of the adjunction is an isomorphism, so this pushout square is actually (also using that the terminal object is preserved):

$$\begin{array}{ccc} X_A & \xrightarrow{\quad m_A \quad} & Y_A \\ \downarrow & & \downarrow{\scriptstyle Z(\gamma_A)} \\ \mathbf{1} & \xrightarrow{Z(\omega)} & Z(P^{(A)}) \end{array}$$

Fix $A \in \mathcal{M}$ and let $Q := Z(P^{(A)})$. For $A \neq M \in \mathcal{M}$, define $\epsilon_M : Q \to Q$ as the unique morphism such that the following diagram commutes:



For $M = A$, we set $\epsilon_A := Id_Q$ (notice that this is the unique morphism which satisfies $\epsilon_A \circ Z(\gamma_A) = Z(\gamma_A)$ and $\epsilon_A \circ \pi = \pi$).
So this gives us an assignment $\mathcal{M} \to \mathcal{K}(Q, Q) : M \mapsto \epsilon_M$. We now show that this is an injection: Assume $\epsilon_M = \epsilon_N$ but $M \neq N$. Then we have $\epsilon_M \circ Z(\gamma)_M = \epsilon_N \circ Z(\gamma)_M$ from which we conclude that the following diagram is a pushout:

$$\begin{array}{ccc} X_M & \xrightarrow[m_M]{} & Y_M \\ \downarrow & & \downarrow \\ \mathbf{1} & \longrightarrow & \mathbf{1} \end{array}$$

But by the previous lemma, we have that the pushout of a chronic is a pullback, so this is a pullback square. But $\mathbf{1} \to \mathbf{1}$ is an isomorphism, thus its pullback m_M is an isomorphism which is not possible by assumption.

$\square$

3.0.5 Exactness of 2-topoi

Proposition 13. *Every 2-topoi $\mathcal{K}$ is regular.*

Proof. That it has finite limits follows because sheafification preserves finite limits (and is preserved under equivalence of categories.
Since equivalence of categories preserves factorisation, we can assume $\mathcal{K} = Sh_2(\mathcal{C}, \mathbf{Cat})$.
Denote by $i : Sh_2(\mathcal{C}) \to Pr_2(\mathcal{C})$ the embedding and by $a : Pr_2(\mathcal{C}) \to Sh_2(\mathcal{C})$ the sheafification.
Let $f \in Sh_2(\mathcal{C})(F, G)$ be a morphism of sheaves. By factorizing it pointwise, we get a factorization $iF \xrightarrow{e} H \xrightarrow{m} iG$ (in $Pr_2(\mathcal{C})$) where e is acute and m is chronic.
We now show that ae is acute (in $Sh_2(\mathcal{C})$). First notice that if $n : L \to R \in Sh_2(\mathcal{C})$ is chronic, then is $in : iL \to iR \in Pr_2(\mathcal{C})$ chronic, indeed: Let $K \in Pr_2(\mathcal{C})$, by adjointness $a \dashv i$, we have that the following diagram commutes:

$$\begin{array}{ccc} Pr_2(\mathcal{C})(K, iL) & \xrightarrow{in\circ -} & Pr_2(\mathcal{C})(K, iR) \\ \downarrow{\cong} & & \downarrow{\cong} \\ Sh_2(\mathcal{C})(aK, L) & \xrightarrow{n\circ -} & Sh_2(\mathcal{C})(aK, R) \end{array}$$

But $n \circ -$ is injective on objects and fully faithfull, therefore the same holds for $in \circ -$ because the isomorphisms are isomorphisms of categories (hence fully faithfull functors which are bijective on objects). Since this holds for any $K \in \mathcal{C}$, we that in is indeed chronic.
To show that ae is acute, we have to show that for each chronic n and for each sheaf F, the following (commuting) diagram is a pullback square:

$$\begin{array}{ccc} Sh_2(\mathcal{C})(aH, L) & \xrightarrow{n\circ -} & Sh_2(\mathcal{C})(aH, R) \\ \downarrow{\circ - ae} & & \downarrow{\circ - ae} \\ Sh_2(\mathcal{C})(F, L) & \xrightarrow{n\circ -} & Sh_2(\mathcal{C})(F, R) \end{array}$$

Since n is chronic, in is chronic (by the previous claim), thus by acuteness of e, the following diagram is a pullback square (in $Pr_2(\mathcal{C})$):

$$\begin{array}{ccc} Pr_2(\mathcal{C})(H, iL) & \xrightarrow{in\circ -} & Pr_2(\mathcal{C})(H, iR) \\ \downarrow{\circ - e} & & \downarrow{\circ - e} \\ Pr_2(\mathcal{C})(iF, iL) & \xrightarrow{in\circ -} & Pr_2(\mathcal{C})(iF, iR) \end{array}$$

But up to isomorphism (by the adjunction $a \dashv i$), these diagrams are the same which shows the claim. □

Corollary 3. *Every 2-topoi $\mathcal{K}$ is exact.*

Proof. It only remains to prove that each congruence is the congruence associated to some arrow. Let $\mathcal{K} = Sh_2(\mathcal{C})$. If $\mathbb{E}$ is a congruence in $\mathcal{K}$, we can consider $\mathbb{E}$ in $Pr_2(\mathcal{C})$ which we know is exact. Therefore we know that $\mathbb{E} = \mathbb{E}(q)$ for some 1-cell q. Since sheafification preserves colimits, we have that this equality translates (under the sheafification) to $\mathcal{K}$ which shows the claim. □

Definition 29. *Coproducts in a 2-category are* ***universal*** *if they are preserves by pullback. If any two distinct coprojections into a coproduct have an initial comma object, then the coproduct is* ***disjoint****.*

Example 20. **Cat** *has (small) universal disjoint coproducts.*

Proof. Let $\mathcal{A}, \mathcal{B} \in \mathbf{Cat}$ and $F : \mathcal{A} \to \mathcal{A} \sqcup \mathcal{B}, G : \mathcal{B} \to \mathcal{A} \sqcup \mathcal{B}$ be the coprojections.
In **Cat**, the comma objects are the comma categories, so the comma object of F and G would be the category where the objects are of the form $(A, B, \alpha : FA \to GB)$ with $A \in \mathcal{A}, B \in \mathcal{B}$ and $\alpha \in \mathcal{A} \sqcup \mathcal{B}$, but such an α can not exist, so the comma object has no objects and is thus the empty category.
We now show that the coproducts are universal. Let $f : \mathcal{A} \to \bigsqcup_{j\in I} \mathcal{B}_j \in \mathbf{Cat}$. Let $\mathcal{P}_i$ be the pullback of f along the i-th coprojection s_i of $\mathcal{B}_i$ into $\bigsqcup_{j\in I} \mathcal{B}_j$, i.e. the following diagram is a pullback square:

$$\begin{array}{ccc} \mathcal{P}_i & \xrightarrow{f_i} & \mathcal{B}_i \\ \downarrow{\scriptstyle t_i} & & \downarrow{\scriptstyle s_i} \\ \mathcal{A} & \xrightarrow{f} & \bigsqcup_{j\in I} \mathcal{B}_j \end{array}$$

Since s_i is a mono, so is t_i, thus we have that $\mathcal{P}_i$ is the full sub-2-category given by:

$$\{A \in \mathcal{A} | f(A) \in \mathcal{B}_i\} .$$

Since the $\mathcal{B}_j$'s are disjoint in $\bigsqcup_{j\in I} \mathcal{B}_j$, we have that the $\mathcal{P}_j$'s are disjoint subcategories of $\mathcal{A}$. Since the $\mathcal{P}_j$'s clearly cover $\mathcal{A}$ (since $f(A) \in \mathcal{B}_i$ for some i), the functor $\bigsqcup_i \mathcal{P}_i \to \mathcal{A}$ is an isomorphism. □

Example 21. *Any* 2*-topos has all small universal and disjoint coproducts.*

Proof. Since **Cat** is exact with all small disjoint universal coproducts, so is $Fun_2(\mathcal{C}^{op}, \mathbf{Cat})$ since limits and colimits are computed pointwise. Since sheafification preserves colimits and finite limits, pullback stability and disjointness of coproducts are preserved. □

3.0.6 Street's theorem

We call a collection of cardinality no greater then the cardinality of $Ob(\mathbf{Set})$ **moderate**.

Theorem 5. *Let $\mathcal{K}$ be a* 2*-category with small hom-categories. The following are equivalent:*

1. *$\mathcal{K}$ is a* 2*-topos.*

2. *$\mathcal{K}$ is lex-total and there exists a moderate set $\mathcal{M} \subseteq Ob(\mathcal{K})$ such that for each $X \in \mathcal{K}$, there exists an acute arrow $M \to X$ with $M \in \mathcal{M}$.*

3. *Every* **Cat***-valued canonical* 2*-sheaf on $\mathcal{K}$ is representable and $\mathcal{K}$ has an acutely generating small set of objects.*

4. *$\mathcal{K}$ is an exact 2-category which has disjoint universal small coproducts and has an acutely generating small set of objects.*

5. *There exists a finitely complete, small canonical 2-site $\mathcal{C}$ and an equivalence $\mathcal{K} \cong Sh(\mathcal{C}, \mathbf{Cat})$.*

Proof. We show the theorem in the following way: $1 \implies 2 \implies 3 \implies 5 \implies 1$ and $1 \implies 4 \implies 3$. That $[5 \implies 1]$ holds is immediate.
$[1 \implies 2]$: That every 2-topos is lex-total is theorem (4). For $\mathcal{M}$ we can take all objects of the underlying site because every presheaf is the colimit of representables and by definition we have that this is a small 2-category.
$[2 \implies 3]$: The first statement is proposition (12) and the second statement is lemma (25).
$[3 \implies 5]$: Since every **Cat**-valued sheaf is representable, Yoneda lemma restrict to an equivalence $\mathcal{K} \cong Sh(\mathcal{K}, \mathbf{Cat})$. Notice that the canonical topology consists of the acute sets. By lemma (22), there exists a small full sub-2-category $\mathcal{C}$ which is closed under finite limits and chronic subobjects and which contains an acutely generating set of objects $\mathcal{G}$. So we the inclusion of $\mathcal{C}$ in $\mathcal{K}$ induces a fully faithfull 2-functor $\mathcal{J}$. Notice that $\mathcal{J}$ is left exact since $\mathcal{C}$ is closed under finite limits. For $X \in \mathcal{K}$, let $\mathcal{U}$ be the set consisting of all arrows into X with domain in $\mathcal{G}$. Since $\mathcal{G}$ is acutely generating we have that $\mathcal{U}$ is acute and thus $\mathcal{U} \in Cov_{\mathcal{K}}(X)$. So if we endow $\mathcal{C}$ with the largest topology such that for the restriction to $\mathcal{C}$ of every sheaf on $\mathcal{K}$ is again a sheaf, the comparison lemma gives us the equivalence $Sh(\mathcal{K}, \mathbf{Cat}) \cong Sh(\mathcal{C}, \mathbf{Cat})$. So we have an equivalence $\mathcal{K} \cong Sh(\mathcal{C}, \mathbf{Cat})$.
$[1 \implies 4]$: That $\mathcal{K}$ has an acutely generating small set of objects follows from $(1 \implies 2 \implies 3)$. The exactness of $\mathcal{K}$ and the pullback stability and disjointness of coproducts is shown in the section of exactness of 2-topoi.
$[4 \implies 3]$: Regard $\mathcal{K}$ as a canonical 2-site. We have to show that

$$R : \mathcal{K} \to Sh(\mathcal{K}, \mathbf{Cat}) : X \mapsto \mathcal{K}(-, X),$$

induced by Yoneda embedding, is an equivalence of 2-categories.
We show that for each $F \in Sh_2(\mathcal{K}, \mathbf{Cat})$, there exist some $X \in \mathcal{K}$ and an acute $\mathcal{K}(-, X) \to F$:
Let $\mathcal{G}$ be a small acutely generating set of objects (of $\mathcal{K}$), i.e. for each $U \in \mathcal{K}$, we have that

$$\mathcal{U} := \{G \to U \mid G \in \mathcal{G}\},$$

is acute. Since R preserves acuteness of small sets, we have that

$$R\mathcal{U} := \{RG \to RU \mid G \in \mathcal{G}\},$$

is acute. Since F is the colimit of representables, we also have that $\{RV \to F \mid V \in \mathcal{K}\}$ is acute. So the *composition* of these two sets

$$\{RG \to RU \to F \mid G \in \mathcal{G}, U \in \mathcal{K}\},$$

is acute. Thus we also have that the following larger set is acute:

$$\{RG \to F \mid G \in \mathcal{G}, U \in \mathcal{K}\}.$$

Since $\mathcal{K}$ has all (small) coproducts, we have that the induced 1-cell (i.e. 2-natural transformation) is acute:

$$\left(\bigsqcup_{G\in\mathcal{G}} RG\right) \to F.$$

But $RG = \mathcal{K}(-, G)$ and R preserves all (small) coproducts, thus we have that

$$\left(\bigsqcup_{G\in\mathcal{G}} RG\right) \to F,$$

is an acute 1-cell. Thus for each 2-sheaf F, we have an acute $\mathcal{K}\left(-, \bigsqcup_{G\in\mathcal{G}} G\right) \to F$. We now show that each $F \in Sh_2(\mathcal{K}, \mathbf{Cat})$ is isomorphic to $\mathcal{K}(-, Z)$ for some $Z \in \mathcal{K}$. Let $e : \mathcal{K}(-, X) \to F$ be an acute 1-cell. We show that its associated congruence $\mathbb{E}(e)$ (on F) is isomorphic to $R\tilde{\mathbb{E}}$ with $\tilde{\mathbb{E}}$ a congruence in $\mathcal{K}$. This is done in three steps:

1. Assume there exists a chronic $k : F \to RY$. Then one can show $\mathbb{E}(e) \cong \mathbb{E}(ke)$. Since
$$\mathcal{K}(-, X) \xrightarrow{e} F \xrightarrow{k} \mathcal{K}(-, Y),$$
we conclude by Yoneda that $ke = Rf$ for some 1-cell f in $\mathcal{K}$. Thus
$$\mathbb{E}(e) \cong \mathbb{E}(ke) \cong \mathbb{E}(Rf) \cong R\left(\mathbb{E}(f)\right).$$

2. Assume there exists a faithful $k : F \to RY$. Then one can show that $\mathbb{E}(e) \to \mathbb{E}(ke)$ is chronic in each component from which one can conclude that $\mathbb{E}(e)$ is isomorphic to a congruence in the image of R.

3. If F is arbitrary, then have the objects of $\mathbb{E}(e)_1 : E_1 \to F_1$ (that is E_1 and F_1) faithful arrows into the image of R using that (F_1, s, t) is a discrete fibration and the image of s and t is RX.

Since $\mathcal{K}$ is exact, it has all quotients, hence we can write $\mathbb{E}(e) = R\left(\tilde{\mathbb{E}}(\tilde{e})\right)$. But R preserves quotients and the quotient of $\mathbb{E}(e)$ has F as its underlying 0-cell. Thus F is indeed representable. $\square$

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