Formal Theory of Rezk completions

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(-2) Homotopy type theory and the univalence axiom

HoTT/UF A foundation for mathematics build upon the notion of (homotopy) types, equipped with points and homotopies (identities).

$$\mathsf{sets} \simeq \mathsf{discrete\ spaces} \hookrightarrow \mathsf{types} \simeq \mathsf{spaces} \simeq \infty - \mathsf{groupoids}$$

$\rightsquigarrow$ constructive, dependent types, proof relevant, ...

Univalence Axiom For every type, the function

$$(X =_{\mathcal{U}} Y) \to (X \simeq Y),$$

is an equivalence of types.

(-1) Semantics

The interpretation of types as spaces/$\infty-$groupoids are proven in:

1. groupoid model (HS, 1996)
2. simplicial set model (KL, 2021)

The category objects in the simplicial set model:

1. segal spaces;
2. univalence $\leftrightarrow$ completeness.

(1) Univalence principle for 1-categories (AKS, 2014)

Let X and Y be categories.

$$\begin{tikzcd}
 & (X \cong Y) \arrow[d, "\mathsf{isotoadjeq}"] \arrow[r, dashed] & \text{adjoint isomorphism} \\
 & (X \simeq Y) \arrow[d, "\mathsf{adjeqtoweq}"] \arrow[r, dashed] & \text{adjoint equivalence} \\
(X = Y) \arrow[uur, "\mathsf{idtoiso}"] \arrow[r] & (X \simeq_w Y) \arrow[r, dashed] & \text{weak equivalence}
\end{tikzcd}$$

Furthermore:

1. Univalence Axiom $\Rightarrow$ $\mathsf{idtoiso}$ is an equivalence of types;
2. X and Y univalent $\Rightarrow$ $\mathsf{isotoadjeq}$ is an equivalence
3. X univalent $\Rightarrow$ $\mathsf{adjeqtoweq}$ is an equivalence

(3) Weak equivalences, point-free

Let X, Y and Z be sufficiently small and $f : X \to Y$ a functor. The precomposition functor $\mathsf{Cat}(f, Z)$ lifts to a category, univalently displayed over $\mathsf{Cat}(X, Z)$:

$$\begin{tikzcd}
 & Act(f,Z) \arrow[d, "\pi_1"] \\
\mathsf{Cat}(Y,Z) \arrow[ur, dashed, "(f \cdot^e_Z -)"] \arrow[r, "(f\cdot -)"'] & \mathsf{Cat}(X,Z)
\end{tikzcd}$$

such that:

1. Z is univalent $\simeq$ $Act(f, Z)$ is univalent for every f;
2. f is fully faithful $\simeq$ π_1 is an isomorphism for every Z;
3. f is essentially surjective $\simeq$ $(f \cdot^e_Z -)$ is an isomorphism, if Z is univalent.

(5) Univalent completion for bicategories, a.k.a. coherence theorem

It is false that every bicategory is equivalent to a univalent one (even with choice).

Theorem (AFMVW, 2022) Every locally univalent bicategory is weak equivalent to a global univalent bicategory.

Theorem Every bicategory is weak (bi)equivalent to a locally/hom-wise univalent bicategory.

Theorem The Rezk completion of Cat is *not* $\mathsf{Cat}_{\mathsf{univ}}$.

$$\begin{tikzcd}
 & \mathsf{RC}^2(\mathsf{Cat}) \arrow[d, dashed, "\exists!"] \\
\mathsf{Cat} \arrow[ur] \arrow[r, "\mathsf{RC}^1"'] & \mathsf{Cat}_{\mathsf{univ}}
\end{tikzcd}$$

The Rezk completion is a fundamental completion in HoTT/UF. Nonetheless, the universal property thereof does not follow from one satisfied by presheaf categories. How to axiomatize/interpret univalent category theory internal to a cosmos (here, Yoneda structures)?

(0) Univalent categories (AKS, 2014)

Definition

A category X is **univalent** if for all objects x and y, the function

$$\mathsf{idtoiso} : (x = y) \to (x \cong y),$$

is an equivalence.

Examples

1. the category of 0-types (sets/discrete spaces) and functions;
2. categories of group objects and group homomorphisms;
3. the category associated to a proset is univalent if and only if it is anti-symmetric;

Counter examples

1. the walking iso category;
2. the category associated to a (non-trivial) group.

(2) Free univalent completion for 1-categories (AKS, 2014)

Universal property if $f : X \to Y$ is a weak equivalence, then

$$\mathsf{Cat}(f, Z) : \mathsf{Cat}(Y, Z) \to \mathsf{Cat}(X, Z),$$

is an isomorphism for every univalent Z.

Construction the corestriction of $\text{よ}_X : X \to [X^{op}, \mathsf{hSet}]$ is a weak equivalence.

Definition The **Rezk completion** for X is a term in type:

$$Rezk(X) := \sum_{\mathsf{RC}(X):\mathsf{Cat}} \; \sum_{h : X \to \mathsf{RC}(X)} \mathsf{is\_weak\_equivalence(h)}.$$

Proposition

1. $Rezk(X)$ is contractible: inhabited and unique (up to identity);
2. $(X \simeq_w Y) \simeq (Rezk(X) \simeq Rezk(Y))$.

Hence:

$$\mathsf{Cat}_{\mathsf{univ}} \hookrightarrow \mathsf{Cat},$$

is reflective.

(4) Construction (SW, 1978)

where

$$Act(f, Z) := \int_{g : X \to Z} \mathsf{Mon}_{\mathsf{Cat}(X, \mathbb{P}X)}(f, g) \quad (:= \mathsf{ExNat}(f, Z))$$

where $\mathsf{Mon}(f, g)$ consists of (suitable) 2-cells:

$$\begin{tikzcd}
 & X \arrow[dl, "f"'] \arrow[dr, "g"] & \\
Y \arrow[rr, Rightarrow, "\theta"] \arrow[dr, "Y(f,1)"'] & & Z \arrow[dl, "Z(g,1)"] \\
 & \mathbb{P}\, X &
\end{tikzcd}$$

(6) Yoneda structure interpretation

Let B be a bicategory, equipped with a coherence system and a Yoneda structure (SW, 1978) with homwise univalent presheaves.

Definition A small object Z is **Rezk** if for any admissible f, $Act(f, Z)$ is univalent.

Proposition The bicategory of univalent objects is univalent.

Theorem Let B be equipped with enough (eso,ff)-factorizations. The corestriction η, of each Yoneda embedding, is the free univalent completion.